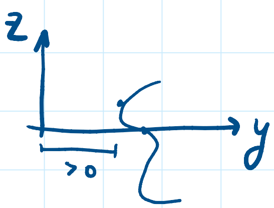
Lecture 13 - Surface Integrals of Scalar Functions cont.

- Read section 7.5 on surface integrals of scalar functions

- Practice Midterm 1 is up on Canvas. I recommend trying it yourself (timed, 90 minutes to complete and scan/upload) before we review it Monday morning.
- Midterm 1 is this upcoming Monday 10/25; the midterm will be available from 12 in the afternoon to 11:59 pm. The midterm will be timed through Gradescope; you will have 90 minutes (as soon as you view the midterm) to complete, scan, and submit your exam. Please start before 10:29 pm to have the full time on the exam.
  - Please correctly assign your solution pages on Gradescope to the corresponding question before submitting your midterm, just as you would do on the homework.

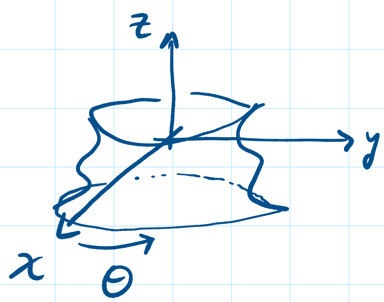
- The morning of the midterm, we will have a review lecture where I will discuss the solutions to the practice midterm. After discussing with some of you, I have decided to give this lecture through Zoom (it will still be live at 8 am on Monday). The Zoom link will be put on Canvas under Zoom LTI PRO; I will also record the session (please remind me if you see that I am not recording) and post the recording to the Media Gallery.
  - This has the benefit that you won't have to commute/walk to the lecture hall and back, which will give you more time to focus/prepare for the exam. Also, it will allow me to stay in the Zoom call after to answer any last minute questions (since normally we would have to leave the lecture hall for the next class that is starting). Consider it an additional office hour.

Surface of Revolution:



Curve parametrized
$y(t),\ z(t) \quad t \in [a,b]$

· Rotate about z-axis, $y(t) > 0$ for all t

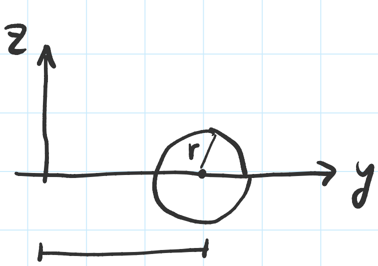


$\Phi(t,\theta) = (y(t)\cos\theta,\ y(t)\sin\theta,\ z(t))$

$\Phi : D \to \Phi(D)$

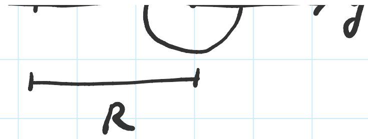
$D = [a,b] \times [0, 2\pi]$

ex/ Torus $\mathbb{T}^2$

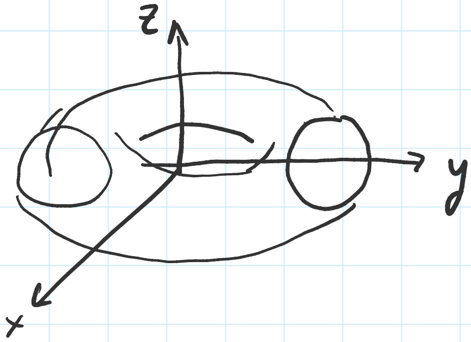


$y(t) > 0 \iff R > r.$

$y(t) = R + r\cos t$



$$y(t) = R + r\cos t \qquad t \in [0, 2\pi]$$
$$z(t) = r\sin t$$



$$\mathbb{T}^2 \simeq \underbrace{\mathbb{S}^1 \times \mathbb{S}^1}_{\text{circles}}$$

$$\Phi(t,\theta) = \big((R + r\cos t)\cos\theta,\ (R + r\cos t)\sin\theta,\ r\sin t\big) \qquad t \in [0, 2\pi],\ \theta \in [0, 2\pi]$$

Compute the surface area of $\mathbb{T}^2$. $\qquad D = [0,2\pi]\times[0,2\pi]$

$$A(\mathbb{T}^2) = \iint_{\mathbb{T}^2} 1\, dS = \iint_D \|\vec{T}_t \times \vec{T}_\theta\|\, dt\, d\theta$$

$\hat{i} \qquad \hat{j} \qquad k$

$$\vec{T}_t = \frac{\partial \Phi}{\partial t} = (-r\sin t\cos\theta,\ -r\sin t\sin\theta,\ r\cos t)$$
$$\vec{T}_\theta = \frac{\partial \Phi}{\partial \theta} = (-(R + r\cos t)\sin\theta,\ (R + r\cos t)\cos\theta,\ 0)$$
$$\vec{T}_t \times \vec{T}_\theta = \big((R + r\cos t)r\cos t\cos\theta,\ -(R + r\cos t)r\cos t\sin\theta,\ -(R + r\cos t)r\sin t\big)$$

$$\|\vec{T}_t \times \vec{T}_\theta\| = \sqrt{(\underbrace{R + r\cos t}_{>0\ (R>r)})^2 r^2} = (R + r\cos t)r.$$

$$A(\mathbb{T}^2) = \int_0^{2\pi}\int_0^{2\pi} (R + r\cos t)r\, dt\, d\theta \qquad \mathbb{T}^2 \simeq \mathbb{S}^1_R \times \mathbb{S}^1_r$$
$$= 2\pi \int_0^{2\pi} (Rr + r^2\cos t)\, dt = 2\pi R \cdot 2\pi r.$$

Surface Integrals of scalar functions

Param: $\Phi : \underset{\subset \mathbb{R}^2}{D} \to \underset{\subset \mathbb{R}^3}{\Phi(D)}$, $\quad f : \Phi(D) \longrightarrow \mathbb{R}$

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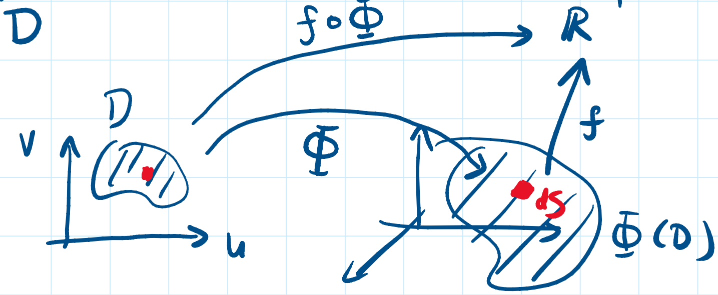
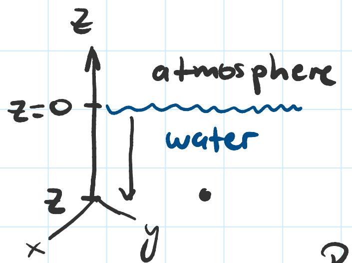
$$\iint_{\Phi(D)} f\, dS = \iint_D f(\Phi(u,v)) \left\| \vec{T}_u(u,v) \times \vec{T}_v(u,v) \right\| du\, dv$$



ex/ Hydrostatic force exerted on a dam
(MAE 101A)



Hydrostatic pressure
$P(x,y,z) = P_0 - \rho g z$
(defined for $z \le 0$)

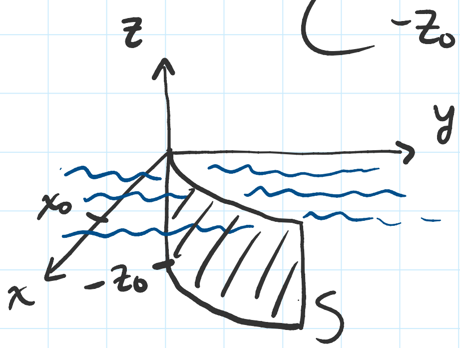
$P_0, \rho, g > 0$ (atm pressure: P_0; density: ρ; gravity: g)

$$\text{Pressure} = \frac{\text{force}}{\text{unit area}}$$

Total hydrostatic force $F = \iint_S P\, dS$

· Suppose we have a dam submerged in water ($z \le 0$)

$$S : \left\{ \begin{array}{ll} y = x^2/2 & \\ 0 \le x \le x_0 & (x_0 > 0) \\ -z_0 \le z \le 0 & (z_0 > 0) \end{array} \right\}$$



$$F = \iint_S P\, dS, \quad P(x,y,z) = P_0 - \rho g z \quad (z \le 0)$$
$$P : S \to \mathbb{R}.$$

$\Phi(x,z) = (x, x^2/2, z)$ $\quad x \in [0, x_0]$, $z \in [-z_0, 0]$

S is the graph of $y = g(x,z) = x^2/2$.

For graphs

$y = g(x,z) = x^2/2.$

For graphs,

$$\|\vec{T}_x \times \vec{T}_z\| = \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial z}\right)^2}$$

$$\Phi(x,z) = (x, g(x,z), z) \qquad \vec{T}_x = \left(1, \frac{\partial g}{\partial x}, 0\right) \qquad \vec{T}_z = \left(0, \frac{\partial g}{\partial z}, 1\right)$$

Total hydrostatic force

$$F = \iint_S P\, dS = \iint_D P(\Phi(x,z))\, \|\vec{T}_x \times \vec{T}_z\|\, dx\, dz$$

$$= \int_{-z_0}^{0} \int_0^{x_0} (P_0 - \rho g z)\sqrt{1+x^2}\, dx\, dz$$

$$= \int_{-z_0}^{0} (P_0 - \rho g z)\, dz \int_0^{x_0} \sqrt{1+x^2}\, dx$$

$$= \left(P_0 z_0 + \rho g \frac{z_0^2}{2}\right) \int_0^{x_0} \sqrt{1+x^2}\, dx.$$

side note:
HW4 Q4

$$\int^2 \sqrt{e^{2u} + e^{4u}}\, du$$

$$= \int e^u \sqrt{1 + e^{2u}}\, du$$

$x = e^u$

$$= \int \sqrt{1+x^2}\, dx$$

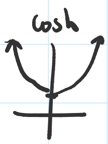
Hyperbolic trig. substitution

$\sqrt{1-x^2}$

$\cos^2\theta + \sin^2\theta = 1$

$i = \sqrt{-1}$

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$



$$\cosh(u) = \frac{e^u + e^{-u}}{2} \qquad \sinh(u) = \frac{e^u - e^{-u}}{2}$$

$$\frac{d}{du}\cosh(u) = \sinh(u), \qquad \frac{d}{du}\sinh(u) = \cosh(u)$$

$$\cosh^2 u - \sinh^2 u = 1$$

$$F = \left(P_0 z_0 + \rho g \frac{z_0^2}{2}\right) \int_0^{x_0} \sqrt{1+x^2}\, dx$$

$x = \sinh(u)$

$dx = \cosh(u)\, du$

$\int^{\sinh^{-1}(x_0)}$

$$dx = \cosh(u)\,du$$

$$= \left(P_0 z_0 + \rho g \frac{z_0^2}{2}\right) \int_0^{\sinh^{-1}(x_0)} \underbrace{\sqrt{1+\sinh^2(u)}}_{=\cosh(u)} \cosh(u)\,du$$

$$= (\dots) \int_0^{\sinh^{-1}(x_0)} \cosh^2(u)\,du$$

$$\cosh^2 u = \frac{e^u + e^{-u}}{2} \cdot \frac{e^u + e^{-u}}{2} = \frac{e^{2u} + 2 + e^{-2u}}{4}$$

$$= \frac{1}{2} + \frac{1}{2}\,\frac{e^{2u} + e^{-2u}}{2} = \frac{1}{2} + \frac{1}{2}\cosh(2u)$$

$$F = (\dots) \int_0^{\sinh^{-1}(x_0)} \left(\frac{1}{2} + \frac{1}{2}\cosh(2u)\right) du$$

$$= (\dots)\left(\frac{1}{2}u + \frac{1}{4}\sinh(2u)\right)\Big|_0^{\sinh^{-1}(x_0)}$$

$$= \left(P_0 z_0 + \frac{1}{2}\rho g z_0^2\right)\left(\frac{1}{2}\sinh^{-1}(x_0) + \frac{1}{4}\sinh(2\sinh^{-1}(x_0))\right)$$

$\square$