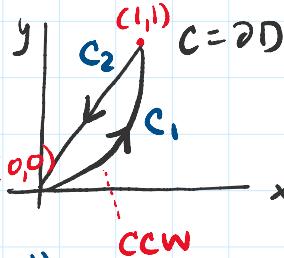
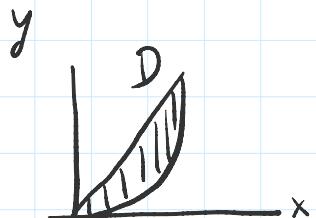


- MT1 grades available to view
- HW6 posted

ex/ let  $\vec{F} = (xy^2, x+y)$

let  $D$  be region

$$\begin{cases} x^2 \leq y \leq x \\ 0 \leq x \leq 1 \end{cases}$$



Verify Green's theorem

$$C_1(x) = (x, x^2), \quad x \in [0, 1]$$

$$C_1'(x) = (1, 2x)$$

$$\underbrace{C_2(s) = (1-s, 1-s^2)}_{\text{CCW}}, \quad s \in [0, 1]$$

$$C_2'(s) = (-1, -2s)$$

$$C_2^-(s) = (s, s) \quad s \in [0, 1]$$

$$\int_C P dx + Q dy = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

$$= \int_0^1 (x(x^2)^2, x+x^2) \cdot (1, 2x) dx = 4/3$$

$$+ \int_0^1 ((1-s)(1-s^2)^2, (1-s)+(1-s^2)) \cdot (-1, -1) ds = -5/4$$

$$= 1/12 \quad \checkmark$$

$$\iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \iint_D \left( \frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial y}(xy^2) \right) dy dx$$

$$= \int_0^1 \int_{x^2}^x (1-2xy) dy dx = \int_0^1 y - xy^2 \Big|_{x^2}^x dx$$

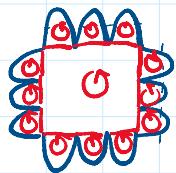
$$= \int_0^1 (x-x^2-x^3+x^5) dx$$

$$= \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{6} = 1/12 \quad \checkmark$$

□

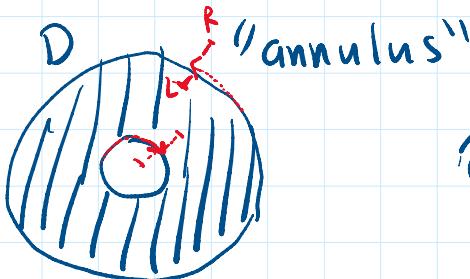
non-simple regions

e.g.

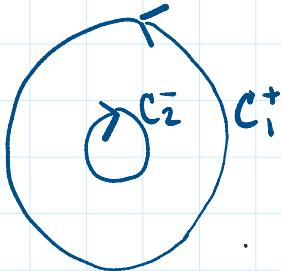


Green's theorem still works  
for non-simple regions

e.g.

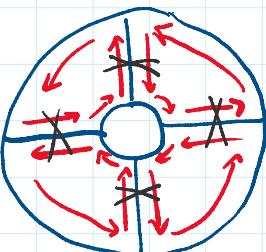


$\partial D$



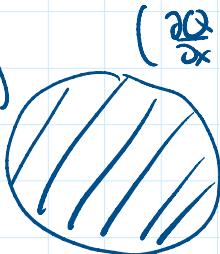
$$\iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \oint_C P dx + Q dy$$

where  $C_1^+ \cup C_2^-$



$$\iint_D (\dots) dA$$

$$=$$



$$-$$



$$\downarrow$$

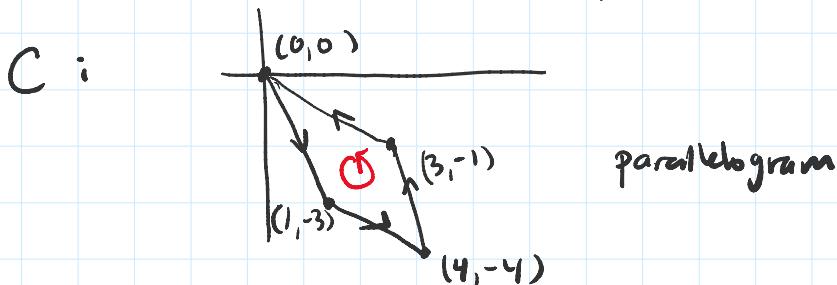
$$= \iint_D \left( P dx + Q dy \right) - \iint_{\text{hole}} \left( P dx + Q dy \right)$$

CW

$$= \iint_D \left( P dx + Q dy \right) + \iint_{\text{hole}} \left( P dx + Q dy \right)$$

ex Evaluate  $\int_C P dx + Q dy$

where  $P(x,y) = -\frac{xy^2}{4}$ ,  $Q(x,y) = \frac{x^2y}{4}$ ,



directly evaluate line integral  $\Rightarrow 4$  line integrals

$$\begin{aligned} \int_C P dx + Q dy &= \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy && D \text{ is area inside } C \\ &= \iint_D \left( \frac{xy}{2} + \frac{xy}{2} \right) dx dy = \iint_D xy dx dy \\ &= (\text{C.O.V. w/ linear transformation}) \\ &= -36 \quad (\text{see MTI problem 1}) \end{aligned}$$

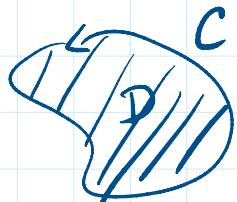
ex Let  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  ( $C^2$  ... twice continuously diff.)

Using Green's theorem, show

$$\oint_C \nabla f \cdot d\vec{r} = 0$$

simple closed curve

proof:  $\nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$



Let D denote the interior region to C

Let  $D$  denote the interior region to  $C$ .

$$\begin{aligned}\oint_C \nabla f \cdot d\vec{r} &= \iint_D \left[ \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) - \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) \right] dx dy \\ &= \iint_D \underbrace{\left[ \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right]}_{=0 \text{ by equality of mixed partials}} dx dy = 0.\end{aligned}$$

□

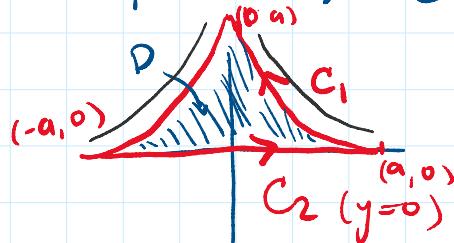
Theorem: Let  $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  ( $C^1$ ),  $F = (P, Q)$

$$\text{Then, } F = \nabla f \text{ if and only if } \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 \quad \square$$

ex, Compute the area of region by the curves

$$(x(\theta), y(\theta)) = (\cos^3 \theta, \sin^3 \theta) \quad \theta \in [0, \pi]$$

and the  $x$ -axis.



$$\text{Area}(D) = \iint_D 1 dx dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \quad \begin{matrix} P=0 \\ Q=x \end{matrix}$$

$$= \int_C P dx + Q dy$$

$$= \int_{C_1} Q dy + \int_{C_2} Q dy$$

$$\begin{matrix} y=0 \leftrightarrow x\text{-axis} \\ \text{---} \\ \text{---} \end{matrix}$$

$\begin{matrix} \leftarrow 0 \\ \text{---} \\ \text{---} \end{matrix}$

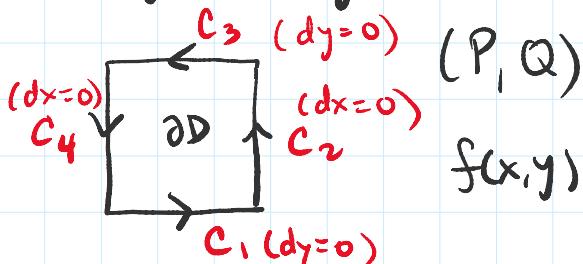
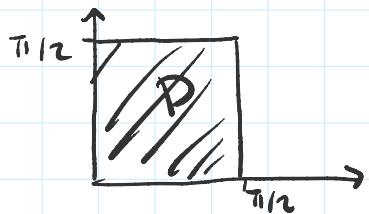
$\begin{matrix} y=\text{const.} \\ \text{---} \\ \text{---} \end{matrix}$

$$\begin{aligned}
 & \int_{C_1} Q dy = \int_{C_1} x dy = \int_0^{\pi} x(\theta) \frac{dy}{d\theta} d\theta \\
 &= \int_0^{\pi} a \cos^3 \theta a \cdot 3 \sin^2 \theta \cos \theta d\theta \\
 &= 3a^2 \int_0^{\pi} \cos^4 \theta \sin^2 \theta d\theta \\
 &= 3a^2 \int_0^{\pi} [\cos^4 \theta - \cos^6 \theta] d\theta \\
 &\quad \cdots (\text{half-angle}) \dots \\
 &= 3a^2 / 16 > 0
 \end{aligned}$$

□

Exercise 8.1.6  
ex/ Verify Green's Theorem

$$P(x,y) = \sin x, \quad Q(x,y) = \cos y \quad D = [0, \pi/2] \times [0, \pi/2]$$



$$f(x,y) = -\cos x + \sin y$$

$$\nabla f = (P, Q)$$

$$\iint_D \left( \underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{=0} \right) dx dy = \iint_D 0 dx dy = 0$$

$$\int_{C=\partial D} P dx + Q dy = \int_{C=\partial D} \sin x dx + \cos y dy$$

$$= \int_{C_1} \cancel{\sin x dx + \cos y dy} + \int_{C_2} \cancel{\sin x dx + \cos y dy} \\ + \int_{C_3} \cancel{\sin x dx + \cos y dy} + \int_{C_4} \cancel{\sin x dx + \cos y dy}$$

$$= \int_0^{\pi/2} \cancel{\sin x dx} + \int_0^{\pi/2} \cancel{\cos y dy} \\ + \int_{\pi/2}^0 \cancel{\sin x dx} + \int_{\pi/2}^0 \cancel{\cos y dy}$$

$$= 0 \quad (\text{alternatively, } (P, Q) = \nabla f \Rightarrow \oint_{\partial D} P dx + Q dy = \oint_{\partial D} \nabla f \cdot d\vec{r} = 0)$$

Green's theorem holds ✓

□

Problem 8 hw6

$$(x(\theta), y(\theta)) \quad \theta \in [0, 2\pi]$$

Include minus sign for this line integral.

