- HW6 due tonight at 11:59 pm.
- I have OH today after lecture and also an additional hour today at 11 am. There are no OH on Thursday due to the Holiday (so me and Soumya's Thursday OH are cancelled for this week).
  - My Thursday OH will be replaced by an OH this Friday at 9 am instead.

- MT2 is this upcoming Monday (11/15), with the same logistical format as MT1. For Monday morning's lecture, we will have a review session where I will go over the practice midterm (remotely through Zoom; Zoom link on Canvas). I will record this lecture as well (please remind me if I'm not recording). I will stay after incase anyone has any last minute questions.
  - The midterm will be available to view on Gradescope from 12 in the afternoon to 11:59 pm. Once you view it, you will have 90 minutes to complete, scan, and upload your exam. Make sure to start before 10:29 pm in order to have the full time, and try to leave 10 minutes at the end to make sure that you have time to upload the exam.
  - The practice midterm is posted on Canvas under the 'Exams and Solutions' module. I recommend doing it yourself before we go over it Monday morning.

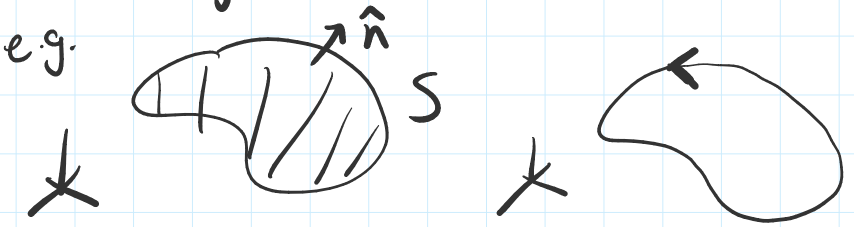
- See the updated course schedule: the last two lectures of the course (12/1 and 12/3) on week 10 will be asynchronous because I will be giving a talk at a conference in Barcelona.
  - My office hours for week 10 will be different; I'll let you all know what my OH will be for that week. I'll try to have more than two OH to make up for the fact that the last two lectures are asynchronous.

Last time:

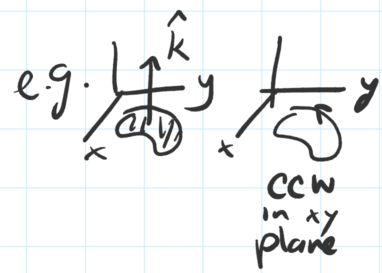
Stokes' Theorem:

Let S be an oriented surface in $\mathbb{R}^3$ and let ∂S denote its oriented boundary; the orientation of the boundary is given by the right hand rule:

e.g.



e.g.



Let $\vec{F}$ be a C^1 vector field. Then,

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}$$

$$\iint_S (\nabla \times F) \cdot d\vec{S} = \int_{\partial S} F \cdot d\vec{r}$$

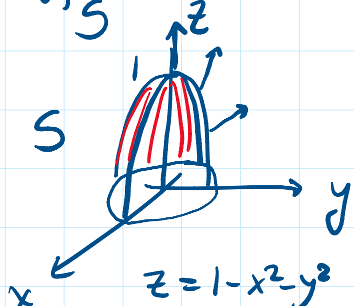
<u>remarks</u>:

- Similar form to FTC I
- If S has no boundary, $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = 0$
  e.g. a sphere 
- What is the boundary of a surface?
  If you're an ant living on S, then ∂S is the set of points you'd leave the surface.

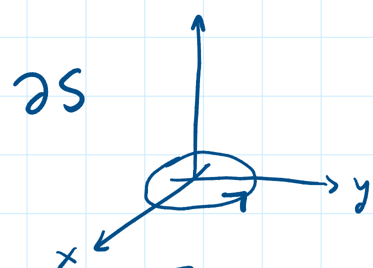
ex/

Let S be the graph of $z = 1 - x^2 - y^2$ where $x^2 + y^2 \leq 1$, with upward normal.
Let $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$, $\vec{F}(x,y,z) = (y, -x, e^{xz})$
Verify Stokes' Theorem

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}$$



$z = 1 - x^2 - y^2$
$x^2 + y^2 \leq 1$



$z = 0$
$x^2 + y^2 = 1$

∂S: $\vec{c}(t) = (\cos t, \sin t, 0) \quad t \in [0, 2\pi]$

$\vec{c}\,'(t) = (-\sin t, \cos t, 0)$

$\vec{F}(\vec{c}(t)) = (\sin t, -\cos t, 1)$

$$\int_{\partial S} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\, dt$$

$$= \int^{2\pi} (-\sin^2 t - \cos^2 t)\, dt = -2\pi.$$

$$= \int_0^{2\pi} (-\sin^2 t - \cos^2 t)\, dt = -2\pi.$$

$S: \quad z = 1 - x^2 - y^2 = g(x,y) \qquad D = \{x^2 + y^2 \le 1\}$

$$\Phi(x,y) = (x, y, g(x,y)) = (x, y, 1 - x^2 - y^2)$$

$$\vec{T}_x \times \vec{T}_y = \left(-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right) = (-2x, -2y, \underbrace{1}_{>0})$$

$$\vec{F} = (y, -x, e^{xz})$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y & -x & e^{xz} \end{vmatrix} = (\cancel{\partial_y e^{xz}} - \cancel{\partial_z(-x)}, -\partial_x e^{xz} + \cancel{\partial_z y}, \partial_x(-x) - \partial_y(y))$$

$$= (0, -z e^{xz}, -2)$$

$$\nabla \times \vec{F}(\Phi(x,y)) = (0, -(1-x^2-y^2)e^{x(1-x^2-y^2)}, -2)$$

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \iint_D (\nabla \times \vec{F})(\Phi(x,y)) \cdot (\vec{T}_x \times \vec{T}_y)\, dx\,dy$$

$D: \ x^2 + y^2 \le 1$

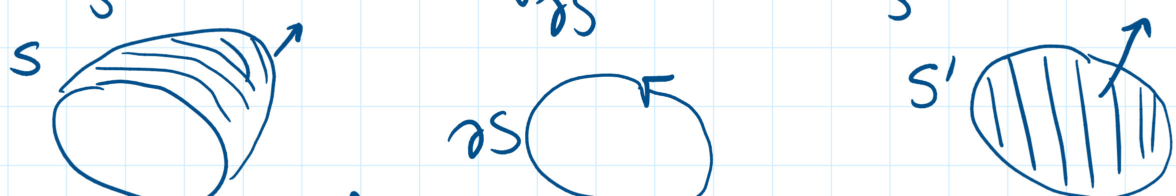
$$= \iint_D \Big(\underbrace{2y(1-x^2-y^2)e^{x(1-x^2-y^2)}}_{\substack{\int(\ldots) = 0 \\ \text{odd w.r.t } y}} - 2 \Big)\, dx\,dy$$

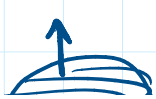
$(x,-y) \in D \Rightarrow (x,y) \in D.$

$$= -2 \iint_D 1\, dx\,dy = -2\,\text{Area}(D) = -2\pi.$$

Let's do this problem 1 more way:

$$\rightarrow \iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{S} = \iint_{S'} (\nabla \times \vec{F}) \cdot d\vec{S}$$





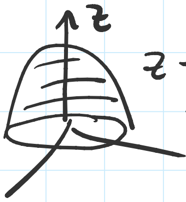
$$\iint (\nabla \times \vec{F}) \cdot d\vec{S}$$

$S_T = S \cup S'_-$

$$0 = \iint_{S_T} (\nabla \times \vec{F}) \cdot d\vec{S}$$
$$= \iint_S (\nabla \times \vec{F}) \, d\vec{S} - \iint_{S'} (\nabla \times \vec{F}) \cdot d\vec{S}$$

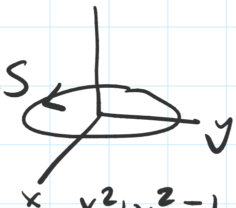
For this ex,

S



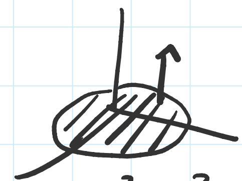
$z = 1 - x^2 - y^2$
$x^2 + y^2 \le 1$

∂S



$x^2 + y^2 = 1$
$z = 0$

S'



$x^2 + y^2 \le 1$
$z = 0$

$$\nabla \times \vec{F} = (0, f(x,y,z), -2) \qquad \hat{n}_{S'} = \hat{k} = (0,0,1)$$

$$\iint_{S'} (\nabla \times \vec{F}) \cdot d\vec{S} = \iint_{S'} (\nabla \times \vec{F}) \cdot \hat{n}_{S'} \, dS$$
$$= -2 \iint_{S'} dS = -2 \, \text{Area}(S') = -2\pi$$

$\square$

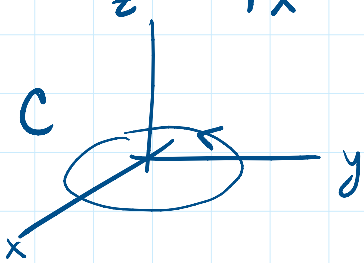
ex/ consider $C : x^2 + y^2 = 1$, $z = 0$, CCW orientation (counterclockwise in the xy plane)

Compute

$$\int_C \vec{F} \cdot d\vec{r}, \quad \vec{F}(x,y,z) = (x^2, y^2 + z, z^2)$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 & y^2 + z & z^2 \end{vmatrix} = (-1, 0, 0)$$

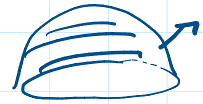
C



Choose S s.t. $\partial S = C$

$z = 1 - x^2 - y^2$

northern hemi
$z = \sqrt{1 - x^2 - y^2}$



disk

Northern Hemisphere: $S = \{x^2+y^2+z^2=1,\ z \geq 0\}$

$$\Phi(\theta,\phi) = (\cos\theta\sin\phi,\ \sin\theta\sin\phi,\ \cos\phi) \qquad \begin{array}{l}\theta\in[0,2\pi]\\ \phi\in[0,\pi/2]\end{array}$$

$$\vec{T}_\phi \times \vec{T}_\theta = \sin\phi\,(\cos\theta\sin\phi,\ \sin\theta\sin\phi,\ \cos\phi)$$

(HW6 problem 1)

$$\int_C \vec{F}\cdot d\vec{r} \overset{\text{Stokes'}}{=} \iint_S (\nabla\times\vec{F})\cdot d\vec{S}$$

$$= \iint_D \Big(\overbrace{\nabla\times\vec{F}\Big|_{\Phi(\theta,\phi)}}^{(-1,0,0)}\Big)\cdot(\vec{T}_\phi\times\vec{T}_\theta)\,d\theta\,d\phi$$

$$= \int_0^{\pi/2}\int_0^{2\pi} \cos\theta\sin^2\phi\,d\theta\,d\phi = 0.$$

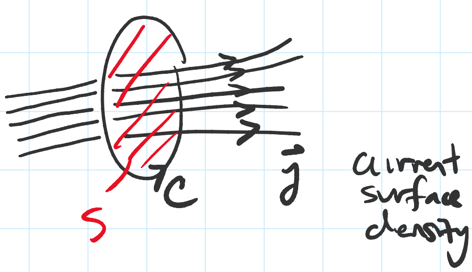
exercise: do the line integral directly, see you get 0.

[NOT TESTED]

Ampere's law in magnetostatics (see end of lecture 10)

$$\oint_C \vec{H}\cdot d\vec{r} = I$$

↳ current passing inside C



$$I = \iint_S \vec{j}\cdot d\vec{S}$$ where S is <u>any</u> surface s.t. $\partial S = C$

Stokes':

$$\iint_{\underline{S}} \underbrace{(\nabla\times\vec{H})}\cdot d\vec{S} = \oint_{\partial S}\vec{H}\cdot d\vec{r} = I = \iint_{\underline{S}} \underbrace{\vec{j}}\cdot d\vec{S}$$

$$\Rightarrow \boxed{\nabla \times \vec{H} = \vec{j}}$$