

Lecture 26 - Conservative Fields

- Read section 8.3
- HW8 due Wednesday 11/24 at 11:59 pm
- CAPEs are now available (for 2 weeks). Please fill them out; I would really appreciate the feedback. Thanks.

Ex/ Let  $S = S_1 \cup S_2$

where  $S_1 = \{(x, y, z) : x^2 + y^2 = 1, z \in [0, 2]\}$ , radially outward normal  
 $S_2 = \{(x, y, z) : x^2 + y^2 \leq 1, z = 2\}$ , upward ( $\hat{z}$ ) normal.

Compute  $\iint_S \vec{F} \cdot d\vec{S}$ ,  $\vec{F}(x, y, z) = (x + y^2 e^z, \cos(x^2) + y + e^{z^2}, z + 1)$

$$\nabla \cdot \vec{F}(x, y, z) = 3$$

$S$  is not a closed surface:

Let  $S_3 = \{(x, y, z) : x^2 + y^2 \leq 1, z = 0\}$  downward ( $-\hat{z}$ ) normal

$$\iint_S \vec{F} \cdot d\vec{S} = \underbrace{\iint_S \vec{F} \cdot d\vec{S}}_{\text{closed surface int.}} + \iint_{S_3} \vec{F} \cdot d\vec{S} - \iint_{S_3} \vec{F} \cdot d\vec{S}$$

$$= \iint_{S \cup S_3} \vec{F} \cdot d\vec{S} - \iint_{S_3} \vec{F} \cdot d\vec{S} \quad \left| \begin{array}{l} W: \\ (x^2 + y^2 \leq 1) \\ z \in [0, 2] \\ \partial W = S \cup S_3 \end{array} \right.$$

$$\stackrel{\text{drv. thm}}{=} \iiint_W (\nabla \cdot \vec{F}) dV - \iint_{S_3} \vec{F} \cdot d\vec{S} \quad \text{geometric formula.}$$

$$= 3 \text{Vol}(W) - \iint_{S_3} \vec{F} \cdot (-\hat{z}) dS$$

$$= 3 \text{Vol}(W) + \iint_{S_3} 1 dS$$

$$= 3 \text{Vol}(W) + \text{Area}(S_3)$$

$$= 3 \cdot \pi \cdot 2 + \pi = 7\pi$$

□

### [Section 8.3] conservative Fields

Theorem:

$$\text{Let } \vec{F} = P \hat{i} + Q \hat{j} + R \hat{k} \text{ under conditions } \dots \text{ in } \mathbb{R}^3$$

Theorem:

Let  $\vec{F}$  be a  $C^1$  vector field on  $\mathbb{R}^3$ .

The following are equivalent:

- (i) For any oriented closed curve  $C$ ,  $\oint_C \vec{F} \cdot d\vec{r} = 0$
- (ii) For any two oriented curves  $C_1$  and  $C_2$  w/ same starting & ending points,

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

- (iii)  $\vec{F} = \nabla f$  for some scalar function  $f: \mathbb{R}^3 \rightarrow \mathbb{R}$
- (iv)  $\nabla \times \vec{F} = 0$ .

proof:

$$(i) \Rightarrow (ii)$$

$$\uparrow \qquad \downarrow$$

$$(iv) \Leftarrow (iii)$$

$$(i) \Rightarrow (ii)$$

$C_1 \cup C_2^-$  is an oriented closed curve

$$0 = \int_{C_1 \cup C_2^-} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2^-} \vec{F} \cdot d\vec{r}$$

$$= \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r}$$



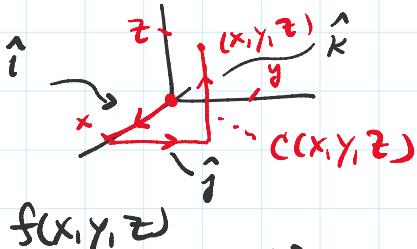
$C_1 \cup C_2^-$   
oriented  
closed  
curve

$$(ii) \Rightarrow (iii)$$

$$f(x, y, z) = \int_{C(x, y, z)} \vec{F} \cdot d\vec{r}$$

where  $C(x, y, z)$  is a curve connecting  $(0, 0, 0)$  to  $(x, y, z)$ .

By (ii), can consider any such curve.

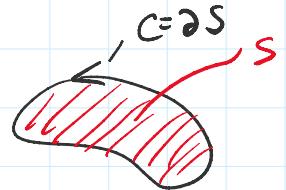


$$\begin{aligned}
 f(x, y, z) &= \int_C \vec{F} \cdot d\vec{r} \\
 &= \int_0^x F_1(t, 0, 0) dt + \int_0^y F_2(x, t, 0) dt + \int_0^z F_3(x, y, t) dt \\
 \frac{\partial f}{\partial z} &= \frac{\partial}{\partial z} \int_0^z F_3(x, y, t) dt \stackrel{\text{FTC II}}{=} F_3(x, y, z)
 \end{aligned}$$

use similar curves for  $\partial f / \partial x, \partial f / \partial y \Rightarrow \nabla f = \vec{F}$

$$(iii) \Rightarrow (iv) \quad \vec{F} = \nabla f \Rightarrow \nabla \times \vec{F} = 0 \quad \text{already did this } \checkmark$$

$$(iv) \Rightarrow (i) \quad \nabla \times \vec{F} = 0 \Rightarrow \oint_C \vec{F} \cdot d\vec{r} = 0$$



$$\begin{aligned}
 \oint_C \vec{F} \cdot d\vec{r} &= \underset{\substack{\uparrow \\ \text{Stokes}}}{\iint_S} (\nabla \times \vec{F}) \cdot d\vec{r} = 0. \\
 &\therefore \partial S = C
 \end{aligned}$$

□

### Theorem:

For a  $C^1$  vector field  $\vec{F}$ , the following are equivalent:

(i) For any oriented closed surface  $S$ ,  $\iint_S \vec{F} \cdot d\vec{S} = 0$

(ii) For any two oriented surfaces  $S_1$  and  $S_2$ ,  
s.t.  $\partial S_1 = \partial S_2$ ,

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_{S_2} \vec{F} \cdot d\vec{S}$$

(iii)  $\vec{F} = \nabla \times \vec{G}$  for some vector field  $\vec{G}$

(iv)  $\nabla \cdot \vec{F} = 0$

□

Remark:

vector spaces

Let  $L: V \rightarrow W$  be a linear operator. //

$$\text{ker}(L) = \{u \in V : Lu = 0\} \subset V$$

$$\text{im}(L) = \{Lu : u \in V\} \subset W$$

notice that functions (on  $\mathbb{R}^3$ ) form a vector space,

$$F(\mathbb{R}^3, \mathbb{R}) \quad (\text{smooth})$$

and also vector fields (on  $\mathbb{R}^3$ ) form a vector space  
 $\mathcal{E}(\mathbb{R}^3, \mathbb{R}^3)$  ( $\text{smooth}$ )

$$\nabla : F(\mathbb{R}^3, \mathbb{R}) \rightarrow \mathcal{E}(\mathbb{R}^3, \mathbb{R}^3)$$

linear operator

$$\nabla_x : \mathcal{E}(\mathbb{R}^3, \mathbb{R}^3) \rightarrow \mathcal{E}(\mathbb{R}^3, \mathbb{R}^3)$$

"

$$\nabla \cdot : \mathcal{E}(\mathbb{R}^3, \mathbb{R}) \rightarrow F(\mathbb{R}^3, \mathbb{R})$$

"

$$\text{ker}(\nabla_x) = \text{im}(\nabla)$$

de Rham cohomology on  $\mathbb{R}^3$

$$\text{ker}(\nabla \cdot) = \text{im}(\nabla_x)$$

ex Let  $\vec{F}(x, y, z) = (2x, e^z \cos(y^2) \cdot 2y, e^z \sin(y^2))$

(i) Prove that  $\vec{F}$  is a gradient vector field, by  
showing  $\nabla \times \vec{F} = 0$

(ii) Find a scalar function  $f$  s.t.  $\vec{F} = \nabla f$

(iii) Compute  $\int_C \vec{F} \cdot d\vec{r}$  where  $C$  is some curve

connecting  $(0, 0, 0)$  to  $(1, 1, 1)$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & e^z \cos(y^2) \cdot 2y & e^z \sin(y^2) \end{vmatrix} = \underbrace{\left( e^z \cos(y^2) \cdot 2y - e^z \cos(y^2) \cdot 2y, 0, 0 \right)}_{=0}$$

$$(F_1, F_2, F_3) = \vec{F} = \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$$

$$f(x, y, z) = \int F_1(x, y, z) dx = x^2 + C_1(y, z)$$

$$f(x, y, z) = \int F_1(x, y, z) dx = x^2 + C_1(y, z)$$

$$f(x, y, z) = \int F_2(x, y, z) dy = e^z \sin(y^2) + C_2(x, z)$$

$$f(x, y, z) = \int F_3(x, y, z) dz = e^z \sin(y^2) + C_3(x, y)$$

$$\Rightarrow f(x, y, z) = x^2 + e^z \sin(y^2).$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} \stackrel{\substack{\uparrow \\ FTI}}{=} f \Big|_{(0,0,0)}^{(1,1,1)} = 1 + e^{\sin(1)} - 0 - 0 \quad \square$$