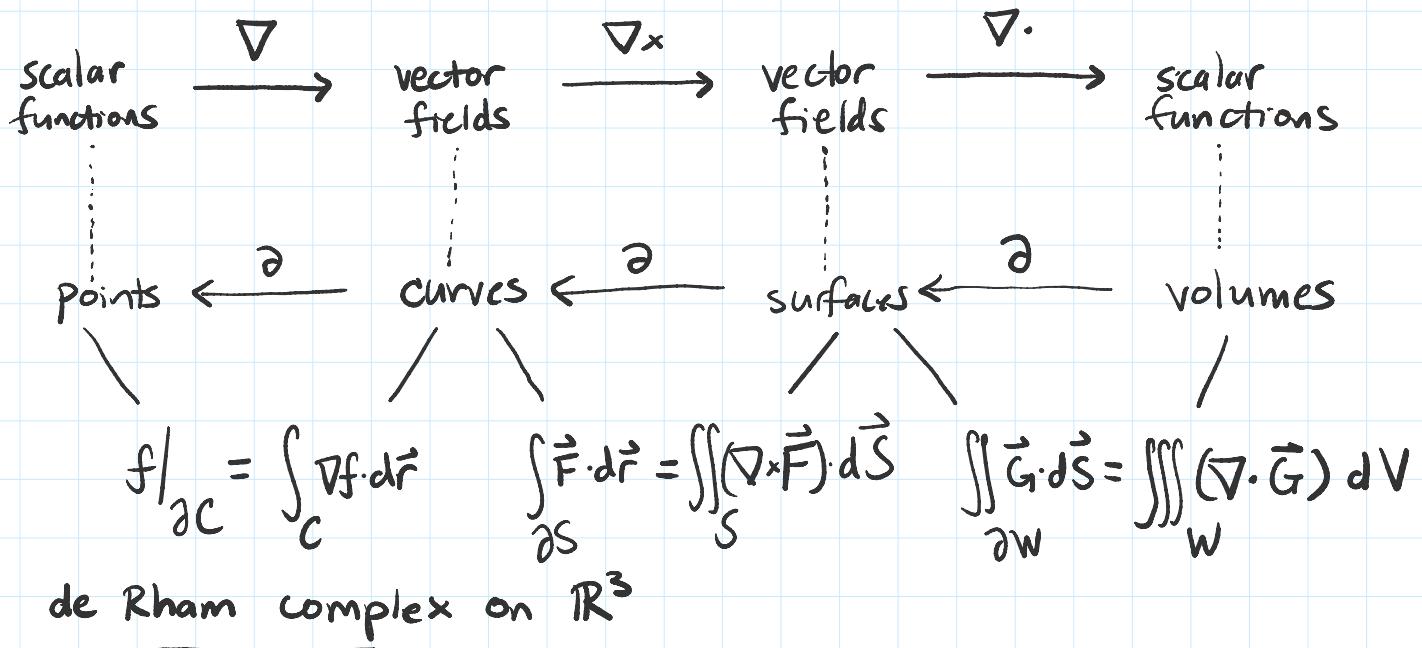


Lecture 27 - Differential Forms on  $\mathbb{R}^3$  (NOT TESTED)

- I have OH today after class today as usual (9 - 10 am). I will not have OH tomorrow due to Thanksgiving.
- HW9 restructured: Homework 9 will not be graded. Furthermore, rather than assigning additional problems, I have decided to just treat the Practice Final as HW9 (which is now posted). My thoughts on this:
  - o The problems on HW1-8 already covered the main material of the course, so HW9 would have been just review problems anyway.
  - o To not overload you all during Thanksgiving break with a homework set and studying for the final, I think the practice final should be enough.
  - o I highly recommend doing the practice final on your own (before I review the solutions in the last two asynchronous lectures). Since you don't have to worry about a homework set, I think this is a reasonable request to ask of you.
  - o If you have any issues with this (for example, if you wanted to turn in another homework set to raise your average homework score), you can e-mail me and we can work something out individually. I don't want to hurt anyone's attempt/possibility to raise their grade.
- Reminder: please fill out your CAPEs. Thanks in advance.
- The Final will take place in-person on Monday December 6th from 8 am to 11 am at WLH 2001 (this lecture hall). You may bring two sheets of hand-written notes (four pages front and back); you may not use any other resources (no calculator, no texts, no communication, etc.).
- Have a happy Thanksgiving! :)



Goal: construct a hierarchy of vector spaces, known as differential forms, which unify the above complex.

Let  $V$  be a vector space.

We say  $\{v_1, \dots, v_k\}$  are linearly independent if  $v_1 \wedge \dots \wedge v_k \neq 0$ .

We say  $\{v_1, \dots, v_k\}$  are linearly independent

$$\text{if: } \alpha_1 v_1 + \dots + \alpha_k v_k = 0 \iff \alpha_1 = 0, \dots, \alpha_k = 0$$

We say  $\{v_1, \dots, v_m\}$  span  $V$  if every  $v \in V$  can

be written as a linear combination of  $\{v_1, \dots, v_m\}$

We say  $\{v_1, \dots, v_n\}$  form a basis for  $V$

if they are lin. ind. and span  $V$ .

$$\dim(V) = n.$$

### Differential Forms on $\mathbb{R}^3$

- all functions are smooth (infinitely differentiable)

0-forms on  $\mathbb{R}^3$ ,  $\Omega^0(\mathbb{R}^3) = \{\text{functions } \mathbb{R}^3 \rightarrow \mathbb{R}\}$

given  $f, g \in \Omega^0(\mathbb{R}^3)$ ,

$$\alpha f + \beta g \in \Omega^0(\mathbb{R}^3) \quad \alpha, \beta \in \mathbb{R} \Rightarrow \Omega^0(\mathbb{R}^3) \text{ vector space.}$$

1-forms on  $\mathbb{R}^3$ ,  $\Omega^1(\mathbb{R}^3)$

Line integral

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \underbrace{F_1 dx + F_2 dy + F_3 dz}_{\Rightarrow \Omega^1(\mathbb{R}^3)}$$

$$\Omega^1(\mathbb{R}^3) = \left\{ F_1 dx + F_2 dy + F_3 dz : F_1, F_2, F_3 \in \Omega^0(\mathbb{R}^3) \right\}$$

We think of  $\{dx, dy, dz\}$  as a basis

$$\begin{aligned} & \alpha(F_1 dx + F_2 dy + F_3 dz) + \beta(G_1 dx + G_2 dy + G_3 dz) \\ &= (\alpha F_1 + \beta G_1) dx + (\alpha F_2 + \beta G_2) dy + (\alpha F_3 + \beta G_3) dz \in \Omega^1(\mathbb{R}^3) \end{aligned}$$

ex/ Differential of a function,  $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ ,  $f(x, y, z)$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \in \Omega^1(\mathbb{R}^3)$$

$$d\sigma = \frac{\partial \sigma}{\partial x} dx + \frac{\partial \sigma}{\partial y} dy + \frac{\partial \sigma}{\partial z} dz \in \Omega^1(\mathbb{R}^3)$$

If  $x, y, z$  are functions of time  $t$ ,

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} && \text{chain rule} \\ &= \nabla f \cdot \vec{c}'(t) \end{aligned}$$

2-forms on  $\mathbb{R}^3$ ,  $\Omega^2(\mathbb{R}^3)$

Area element  $dx dy$

Surfaces, need to keep track of orientation  
Wedge product:

$$\wedge: \Omega^1(\mathbb{R}^3) \times \Omega^1(\mathbb{R}^3) \longrightarrow \Omega^2(\mathbb{R}^3)$$

$dx \wedge dy$  is the surface area element for a parallelogram with edges  $dx, dy$ , oriented in the  $\hat{k}$  direction



$\Omega^2(\mathbb{R}^3)$

$$= \{ F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy : \begin{matrix} F_1, F_2, F_3 \\ \text{scalar functions} \end{matrix} \}$$

- $dy \wedge dx$  corresponds to surface integration in the  $-\hat{k}$  direction  $\Rightarrow dy \wedge dx = -dx \wedge dy$
- Why not  $dx \wedge dx, dy \wedge dy, dz \wedge dz$ ?

They are all 0,

$$dx \wedge dx = -dx \wedge dx$$

$$2 dx \wedge dx = 0$$

$$\left| \begin{array}{l} dx \wedge dx = -dx \wedge dx \\ 2dx \wedge dx = 0 \\ dx \wedge dx = 0 \end{array} \right.$$

Basis  $\{dy \wedge dz, dz \wedge dx, dx \wedge dy\}$ .

$$\Omega^3(\mathbb{R}^3)$$

$$= \{ f dx \wedge dy \wedge dz : f \text{ scalar function} \}$$

$$\underbrace{dz \wedge dy \wedge dx}_{=0} = -\underbrace{dz \wedge dx \wedge dy}_{=0} = -dx \wedge dy \wedge dz$$

$$\Omega^p(\mathbb{R}^3) \quad p \geq 4 \\ \rightsquigarrow \{0\}$$

$$\Omega^p(\mathbb{R}^n) \\ \text{nonzero } p \leq n.$$

$$\underline{dx \wedge dy \wedge dz \wedge dx}$$

In  $\mathbb{R}^3$ :

$$\Omega^0(\mathbb{R}^3) \quad \text{scalar functions}$$

$$\Omega^1(\mathbb{R}^3) \ni F_1 dx + F_2 dy + F_3 dz \quad \nwarrow \\ \sim F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k} \in \text{Vector Fields on } \mathbb{R}^3$$

$$\Omega^2(\mathbb{R}^3) \ni F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy \\ \sim F_1 \hat{i} \wedge \hat{j} + F_2 \hat{j} \wedge \hat{k} + F_3 \hat{k} \wedge \hat{i} \\ \in \text{Vector Fields on } \mathbb{R}^3$$

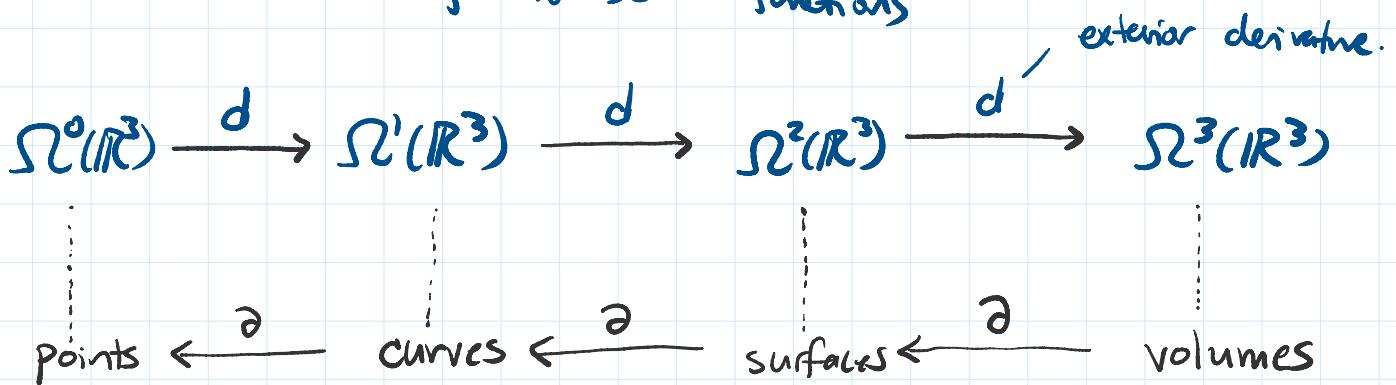
under this identif.  
of 1-forms  $\sim$  v.f.  
 $\Lambda = x$ .

$$\dim \Omega^p(\mathbb{R}^n) = \binom{n}{p} \quad \Omega^0(\mathbb{R}^n) \quad \Omega^1(\mathbb{R}^n) \quad \Omega^2(\mathbb{R}^n) \quad \Omega^3(\mathbb{R}^n) \dots$$

$\dim \Omega^p(\mathbb{R}^n)$	$ _{\text{point}}$	$= \binom{n}{p}$	$\Omega^0(\mathbb{R}^n)$	$\Omega^1(\mathbb{R}^n)$	$\Omega^2(\mathbb{R}^n)$	$\Omega^3(\mathbb{R}^n)$	$\dots$
			$n=0$	$1$			
			$n=1$	$1$		$1$	
			$n=2$	$1$	$2$	$1$	
			$n=3$	$1$	$3$	$3$	$\frac{1}{4}$
			$n=4$	$1$	$4$	$6$	$\frac{1}{4}$

$$\Omega^3(\mathbb{R}^3) \ni f dx \wedge dy \wedge dz$$

$\sim f \sim \text{scalar functions}$



Integrating 3-forms

$$f dx \wedge dy \wedge dz \in \Omega^3(\mathbb{R}^3)$$

W domain/vol in  $\mathbb{R}^3$

$$\left[ \int_W f dx \wedge dy \wedge dz = \iiint_W f dx dy dz \right]$$

Integrating 2-forms

$$\int_S F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy$$

parametrize  $(x(u,v), y(u,v), z(u,v)) = \bar{\Phi}(u,v)$ ,  $\bar{\Phi}(D) = S$ ,

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv$$

$$\dots / \partial v \dots \partial v \dots \partial u \dots \partial u$$

$$\begin{aligned}
 dx \wedge dy &= \left( \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \right) \wedge \left( \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv \right) \\
 &= \frac{\partial x}{\partial u} \frac{\partial y}{\partial u} du \wedge du + \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} du \wedge dv \\
 &\quad + \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} dv \wedge du + \frac{\partial x}{\partial v} \frac{\partial y}{\partial v} dv \wedge dv \\
 &= \left( \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right) du \wedge dv \\
 &= \frac{\partial(x,y)}{\partial(u,v)} du \wedge dv
 \end{aligned}$$

$$\begin{aligned}
 \int_S F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy \\
 &= \int_{\substack{S \\ R^2 \ni D}} \left[ F_1 \frac{\partial(y,z)}{\partial(u,v)} + F_2 \frac{\partial(z,x)}{\partial(u,v)} + F_3 \frac{\partial(x,y)}{\partial(u,v)} \right] du \wedge dv \\
 &= \iint_D (F_1, F_2, F_3) \cdot \underbrace{\left( \frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)} \right)}_{\text{Section 7.4}} du dv \\
 &= \iint_D \vec{F} \cdot (\vec{T}_u \times \vec{T}_v) du dv \\
 &= \iint_S \vec{F} \cdot d\vec{S}
 \end{aligned}$$