Differentiability (2.6); Quiz 1

Differentiability requires continuity. A function \( f \) is differentiable at \( x = a \) if
\[
\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
\]
exists. The value of the derivative at \( x = a \) is equal to the value of the limit, i.e.,
\[
\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = f'(a),
\]
and is equal to the slope of the tangent line of \( f \) at \( x = a \). A function fails to be differentiable at a point when this limit does not exist.

**Example:** Let \( y = \frac{x^2}{x} \). DRAW graph. Note \( y \) is discontinuous at \( x = 0 \). Therefore, \( y \) is not differentiable when \( x = 0 \), because \( \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} \) does not exist. WHY?

A function must be continuous at a point if the function has a derivative at that point. *But is continuity enough for a function to be differentiable?*

**Continuity does not imply differentiability.**

**Example:** DRAW graph of \( y = |x| \). IS this function continuous and differentiable at \( x = 0 \)? The LH limit of the derivative function is equal to
\[
\lim_{h \to 0^-} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0^-} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1.
\]
But the RH limit of the derivative function is equal to
\[
\lim_{h \to 0^+} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0^+} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1.
\]
Therefore, the limit of the derivative function at \( x = 0 \), \( f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} \), does not exist. DRAW graph. Therefore, \( y \) is continuous but not differentiable.

IS the following function continuous and differentiable at \( x = 1 \)?
\[
f = \begin{cases} 
  x, & \text{for } x < 1 \\
  x^2, & \text{for } x \geq 1
\end{cases}
\]

DRAW graph. SHOW \( f \) is continuous at \( x = 1 \), i.e., show the LH and RH limits of \( f \) are both equal to 1. But for \( f \) to be differentiable at \( x = 1 \) the *slopes* on the left and the right must be equal at \( x = 1 \). Clearly, the slope of the LH side is equal to 1; but the slope of the RH side is equal to 2 (SHOW using definition of derivative at a point.) Therefore, \( f \) is continuous but not differentiable.

DRAW graph of \( y = x^{1/3} \). The graph appears to have a vertical tangent at the point \((0, 0)\). SHOW the limit does not exist. Therefore, the function does not have a derivative at \( x = 0 \).

**WHAT** values of \( a \) and \( b \) make the following function continuous and differentiable everywhere? \( g = \begin{cases} 
  ax + 2, & \text{for } x < 0 \\
  b(x - 1)^2, & \text{for } x \geq 0
\end{cases} \)

To be continuous, we must have \( a \cdot 0 + 2 = b(0 - 1)^2 \). So \( b = 2 \). To be differentiable, the slopes must be equal when \( x = 0 \). The slope of the LH side is equal to \( a \); the slope of the RH side is equal to -4 (SHOW using definition of derivative at a point.) Therefore, \( a = -4 \), \( b = 2 \).