The Exponential Function (3.2)

Derivative of $a^x$. DRAW graph of $y = a^x$, $a > 1$. Since the graph is increasing and concave up, $y'$ must be positive and increasing. Since the graph of $y$ is positive and increasing, the graph of $y'$ should be similar. We will show that the function $y' = ky$ for some constant $k$; and, therefore, the graph of $y'$ is a vertical dilation of the graph of $y$ for any exponential function $y = a^x$. DRAW examples.

LET $g(x) = 2^x$. FIND $g'(x)$. Let's calculate $g'(x)$ using the limit definition.

$$g'(x) = \lim_{h \to 0} \frac{2^{x+h} - 2^x}{h} = \lim_{h \to 0} \frac{2^x \cdot 2^h - 2^x}{h} = \lim_{h \to 0} 2^x \cdot \frac{2^h - 1}{h}$$

So, $g'(x)$ is equal to $g(x)$ times the value of $\lim_{h \to 0} \frac{2^h - 1}{h}$ which is independent of $x$! Table 3.2 (p. 118) estimates the value of $\lim_{h \to 0} \frac{2^h - 1}{h}$ by calculating the difference quotient for very small values of $h$. WRITE Table 3.2. We estimate $\lim_{h \to 0} \frac{2^h - 1}{h} \approx 0.693$. Therefore, $g'(x) = 0.693g(x) = 0.693(2^x)$. DRAW $g(x)$ and $g'(x)$.

In general, for $f(x) = a^x$, $f'(x) = a^x \cdot \lim_{h \to 0} \frac{a^h - 1}{h}$. Table 3.3 (p. 118) estimates the values of $\lim_{h \to 0} \frac{a^h - 1}{h}$ for $a = 2, 3, 4, 5, 6, 7$. WRITE Table 3.3. Therefore, we estimate the following derivative functions. DRAW.

<table>
<thead>
<tr>
<th>$f(x) = a^x$</th>
<th>$a = 2$</th>
<th>$a = 3$</th>
<th>$a = 4$</th>
<th>$a = 5$</th>
<th>$a = 6$</th>
<th>$a = 7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f'(x)$</td>
<td>0.693f</td>
<td>1.099f</td>
<td>1.396f</td>
<td>1.609f</td>
<td>1.792f</td>
<td>1.946f</td>
</tr>
</tbody>
</table>

Derivative of $e^x$. It appears that the constant multiplier of $f$ increases as the base $a$ increases. We may guess that for some value between 2 and 3, that the multiplier is exactly one and that $f'(x) = f(x)$. On p. 118, our text shows that this number is $e \approx 2.718\ldots$. This means that $\frac{d}{dx}(e^x) = e^x = e^x$. $y = e^x$ is the ONLY function that is equal to its own derivative! WHAT is the second derivative of $y = e^x$? Ans: $y'' = e^x$.

Derivative of $a^x$. Our text shows on p. 119 that $\lim_{h \to 0} \frac{a^h - 1}{h} = \ln a$. Therefore,

$$\ln 2 \approx 0.693, \ln 3 \approx 1.099 \text{ and so on.}$$

And for $f(x) = a^x$, $f'(x) = a^x \cdot \lim_{h \to 0} \frac{a^h - 1}{h} = (\ln a) a^x$.

For $f(x) = 2^x$, $f'(x) = \ln 2 \cdot (2^x)$. For $g(x) = e^x$, $g'(x) = (\ln e) e^x = 1 e^x = e^x$.

SOLVE 3.2.5.

SOLVE 3.2.37. WHAT are the units of $\frac{dP}{dt}$? WHAT does the negative sign indicate about the population, $P$?

Tangent Lines.

SOLVE 3.1.53.
SOLVE 3.2.44. Let's first find the equation of the tangent line to the graph of $e^x$ at $x = 0$. Since $y' = e^x$, we know that the slope of the tangent line is $y'(0) = e^0 = 1$. Since the graph of $e^x$ passes through the point $(0, 1)$, the equation of the tangent line is $y = x + 1$. WHY? DRAW graphs of $e^x$ and $y = x + 1$. The graph seems to indicate that $e^x \geq x + 1$. This is true, because the exponential graph of $e^x$ is concave up, and we know that "concave up" graphs lie above any tangent line. USE the fact that $e^x$ is concave up to prove this!