**The Product and Quotient Rules (3.3)**

**Product Rule.** WHAT is the derivative of the product of two functions, i.e., 
\[
\frac{d}{dx} [f(x) \cdot g(x)]? \text{ We might guess that it is equal to the product of the two derivatives, i.e., } f'(x) \cdot g'(x). \text{ This is not unreasonable since } \frac{d}{dx} [f(x) \pm g(x)] = f'(x) \pm g'(x). \text{ Let's see if our formula works. FIND } \frac{d}{dx}[(x+2) \cdot (2x-1)]. \text{ If our formula is correct, then this derivative should be equal to } \frac{d}{dx}[(x+2) \cdot \frac{d}{dx}(2x-1)] = 1 \cdot 2 = 2. \text{ However,}
\]
\[
\frac{d}{dx}[(x+2) \cdot (2x-1)] = \frac{d}{dx}(2x^2 + 3x - 2) = 4x + 3, \text{ which is NOT EQUAL to 2. Therefore, our formula } \frac{d}{dx} [f(x) \cdot g(x)] = f'(x) \cdot g'(x) \text{ is incorrect and } \frac{d}{dx} [f(x) \cdot g(x)] \neq f'(x) \cdot g'(x).
\]

Let's try a different approach. Let's try going back to the original definition of the derivative of a function, 
\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.
\]
The derivative of a product of two functions is 
\[
\frac{d}{dx} [f(x) \cdot g(x)] = \lim_{h \to 0} \frac{\Delta f \cdot g(x) + f(x) \cdot \Delta g + \Delta f \cdot \Delta g}{h}.
\]
Let's break this limit up into a sum of three separate limits,
\[
\frac{d}{dx} [f(x) \cdot g(x)] = \lim_{h \to 0} \frac{\Delta f \cdot g(x)}{h} + \lim_{h \to 0} \frac{f(x) \cdot \Delta g}{h} + \lim_{h \to 0} \frac{\Delta f \cdot \Delta g}{h}.
\]
The first limit,
\[
\lim_{h \to 0} \frac{\Delta f \cdot g(x)}{h} = \lim_{h \to 0} g(x) \cdot \frac{\Delta f}{h} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \cdot g(x) = f'(x)g(x).
\]
Similarly, the second limit,
\[
\lim_{h \to 0} \frac{f(x) \cdot \Delta g}{h} = f(x)g'(x).
\]
The third limit,
\[
\lim_{h \to 0} \frac{\Delta f \cdot \Delta g}{h} = \lim_{h \to 0} \frac{\Delta f \cdot \Delta g}{h} \cdot \frac{\Delta f}{h} = 0 \cdot f'(x) \cdot g'(x) = 0. \text{ Therefore, the sum of the three limits is }
\]
\[
\frac{d}{dx} [f(x) \cdot g(x)] = \lim_{h \to 0} \frac{\Delta f \cdot g(x) + f(x) \cdot \Delta g + \Delta f \cdot \Delta g}{h} = f'(x)g(x) + f(x)g'(x). \text{ In other words, } (fg)' = f'g + fg'.
\]

SOLVE 3.1.29. \( f(y) = 4y(2 - y^2) \).

SOLVE 3.3.9. \( y = \sqrt{x}(x + 1) \). Previously we distributed the square root to get \( y = x^{3/2} + x^{1/2} \) and used the Power Rule to get \( y' = \frac{3}{2}x^{1/2} + \frac{1}{2}x^{-1/2} \). Now we will use the
Product Rule to get 
\[ y' = \left(\sqrt{x}\right)' \cdot (x+1) + (x+1)' \cdot \sqrt{x} = \frac{1}{2} x^{-1/2} (x+1) + \sqrt{x}. \]
SHOW that this is equivalent to 
\[ y' = \frac{3}{2} x^{1/2} + \frac{1}{2} x^{-1/2}. \]

Quotient Rule. Suppose we want to differentiate a function of the form 
\[ Q(x) = \frac{f(x)}{g(x)}, \] where \( g(x) \neq 0 \). We want to find a formula for \( Q'(x) \). We can use the Product Rule here. Multiply both sides of the equation by \( g(x) \) to get 
\[ f(x) = Q(x)g(x). \] Now differentiate both sides of the equation to get 
\[ f'(x) = Q'(x)g(x) + Q(x)g'(x). \] Now we solve for \( Q'(x) \), 
\[ Q'(x) = \frac{f'(x) - Q(x)g'(x)}{g(x)}. \] Substituting \( Q(x) = \frac{f(x)}{g(x)} \) into the equation we get 
\[ Q'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}. \] In other words,
\[ \left(\frac{f}{g}\right)' = \frac{fg' - fg'}{g^2}. \]

SOLVE 3.3.22. 
\[ y = \frac{\sqrt{t}}{t^2 + 1}. \]

SOLVE 3.3.21.