The Chain Rule and Inverse Functions (3.6)

Review inverse functions and Chain Rule. Recall that for inverse functions

\[ f^{-1}(f(x)) = f^{-1}(f(x)) = x. \]

And, the Chain Rule states

\[ \frac{d}{dx} \left( g(f(x)) \right) = f'(x) \cdot g'(f(x)). \]

Derivative of \( \sqrt{x} \).

We have already used the Power Rule to find the derivative of \( f(x) = \sqrt{x} \),

\[ f'(x) = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}. \]

We can also use the Chain Rule to find its derivative.

Using the Chain Rule, we can write

\[ \frac{d}{dx} \left( (\sqrt{x})^2 \right) = 2\sqrt{x} \cdot \frac{d}{dx} (\sqrt{x}), \]

where

\[ g(x) = x^2 \quad \text{and} \quad f(x) = \sqrt{x}. \]

But \( \frac{d}{dx} \left( (\sqrt{x})^2 \right) = \frac{d}{dx} (x) = 1. \) Therefore, \( 2\sqrt{x} \cdot \frac{d}{dx} (\sqrt{x}) = 1. \)

Solving for \( \frac{d}{dx} (\sqrt{x}) \), we get \( \frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}}. \)

Derivative of \( \ln x \).

Since \( e^{\ln x} = x \), we must have \( \frac{d}{dx} \left( e^{\ln x} \right) = \frac{d}{dx} (x) = 1. \) Using the Chain Rule,

\[ \frac{d}{dx} \left( e^{\ln x} \right) = \frac{d}{dx} (\ln x) \cdot e^{\ln x} = \frac{d}{dx} (\ln x) \cdot x, \]

where \( g(x) = e^x \) and \( f(x) = \ln x \). Therefore,

\[ \frac{d}{dx} (\ln x) \cdot x = 1 \quad \text{and} \quad \frac{d}{dx} (\ln x) = \frac{1}{x}. \]

SOLVE 3.6.9. \( j(x) = \ln(e^{ax} + b) \). Using the Chain Rule,

\[ \frac{d}{dx} \left( \ln(e^{ax} + b) \right) = \frac{d}{dx} (e^{ax} + b) \cdot \frac{1}{e^{ax} + b} = a e^{ax} \cdot \frac{1}{e^{ax} + b} = \frac{ae^{ax}}{e^{ax} + b}, \]

where \( g(x) = \ln x \) and \( f(x) = e^{ax} + b \).

FIND the tangent line to \( y = \ln(x^2 - 3) \) at \( (2,0) \). \( y' = 2x \cdot \frac{1}{x^2 - 3} = \frac{2x}{x^2 - 3} \),

where \( g(x) = \ln x \) and \( f(x) = x^2 - 3 \). So \( y'(2) = \frac{2 \cdot 2}{2^2 - 3} = \frac{4}{4 - 3} = 4 \), and the tangent line at \( (2,0) \) is \( y = 4(x - 2) + 0 = 4x - 8 \).

Derivative of \( a^x \).

Since \( \ln(a^x) = x \ln a \), we must have \( \frac{d}{dx} \left( \ln(a^x) \right) = \frac{d}{dx} (x \ln a) = \ln a. \) Using the Chain Rule,

\[ \frac{d}{dx} \left( \ln(a^x) \right) = \frac{d}{dx} (a^x) \cdot \frac{1}{a^x}, \]

where \( g(x) = \ln x \) and \( f(x) = a^x \). Therefore,

\[ \frac{1}{a^x} \cdot \frac{d}{dx} (a^x) = \ln a \quad \text{and} \quad \frac{d}{dx} (a^x) = (\ln a)(a^x). \]

FIND \( \frac{d}{dx} (2\sin x). \)

\[ \frac{d}{dx} (2\sin x) = \cos x \cdot (\ln 2) \cdot 2\sin x, \]

where \( g(x) = 2^x \) and \( f(x) = \sin x. \).
Derivative of \( \sin^{-1} x \).

Since \( \sin(\sin^{-1} x) = x \), we must have \( \frac{d}{dx} \left[ \sin(\sin^{-1} x) \right] = \frac{d}{dx} (x) = 1 \). Using the Chain Rule, \( \frac{d}{dx} \left[ \sin(\sin^{-1} x) \right] = \frac{d}{dx} (\sin^{-1} x) \cdot \cos(\sin^{-1} x) \), where \( g(x) = \sin x \) and \( f(x) = \sin^{-1} x \). So \( \frac{d}{dx} (\sin^{-1} x) \cdot \cos(\sin^{-1} x) = 1 \) and \( \frac{d}{dx} (\sin^{-1} x) = \frac{1}{\cos(\sin^{-1} x)} \). But what is \( \cos(\sin^{-1} x) \) equal to? Recall that \( \sin^{-1} x \) represents an angle, say \( \theta \), whose sine is equal to the number \( x \). DRAW. Using the Pythagorean Theorem, we can calculate that the remaining side is equal to \( \sqrt{1-x^2} \). Therefore, the \( \cos(\sin^{-1} x) = \cos \theta = \sqrt{1-x^2} \).

Finally, \( \frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \).

SOLVE 3.6.24. \( \frac{d}{dx} \left[ \cos(\sin^{-1} x) \right] = \frac{d}{dx} (\sin^{-1} x) \cdot [\sin(\sin^{-1} x)] = \frac{1}{\sqrt{1-x^2}} \cdot [\sin(\sin^{-1} x)] \), where \( g(x) = \cos x \) and \( f(x) = \sin^{-1} x \). But \( \sin(\sin^{-1} x) = x \), so \( \frac{d}{dx} \left[ \cos(\sin^{-1} x) \right] = -\frac{x}{\sqrt{1-x^2}} \).

SOLVE 3.6.54. Given that \( f(x) = x^3 \).

a. Find \( f'(2) \). \( f'(x) = 3x^2 \), \( f'(2) = 3 \cdot 2^2 = 12 \).

b. Find \( f^{-1}(x) \). \( f^{-1}(x) = \sqrt[3]{x} = x^{1/3} \).

c. Use your answer from part (b) to find \( (f^{-1})'(8) \). Since \( (f^{-1})'(x) = \frac{d}{dx} x^{1/3} = \frac{1}{3} x^{-2/3} \), \( (f^{-1})'(8) = \frac{1}{3} (8)^{-2/3} = \frac{1}{3} (8^{1/3})^{-2} = \frac{1}{3} (2)^{-2} = \frac{1}{3} \left( \frac{1}{2} \right)^2 = \frac{1}{3} \left( \frac{1}{4} \right) = \frac{1}{12} \).

d. How could you have used your answer from part (a) to find \( (f^{-1})'(8) \)? Since \( f \left( f^{-1}(x) \right) = x \), we can use the Chain Rule to take derivatives of both sides, \( (f^{-1})'(x) \cdot f' \left( f^{-1}(x) \right) = 1 \) or \( (f^{-1})'(x) = \frac{1}{f' \left( f^{-1}(x) \right)} \). So \( (f^{-1})'(8) = \frac{1}{f'(f^{-1}(8))} = \frac{1}{f'(8)} = \frac{1}{f'(2)} = \frac{1}{12} \).