Conic Sections (Conics)

Conic sections are the curves formed when a plane intersects the surface of a right cylindrical double cone.

An example of a double cone is the 3-dimensional graph of the equation
\[ z^2 = x^2 + y^2 \] (or equivalently the two graphs \( z = \pm \sqrt{x^2 + y^2} \).)

Planes in 3-dimensions are graphs of equations of the form \( ax + by + cz + d = 0 \).

An example of a plane in 3-dimensions is \( z = 2 \). This plane is parallel to the xy-plane.

Let's find the intersection of the double cone \( z^2 = x^2 + y^2 \) and the plane \( z = 2 \).
Substituting \( z = 2 \) into \( z^2 = x^2 + y^2 \), we get \( x^2 + y^2 = 4 \) which we recognize as a circle of radius 2 whose center is the point in space, \((0,0,2)\).

In general, the intersections of double cones with planes in space are equations of the form \( Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \); this is called the general quadratic form.

Next, to the general linear form, \( Ax + By + C = 0 \), this is one of the simplest forms of an equation. The graphs of these equations represent the conic sections (circles, parabolas, ellipses, and hyperbolas), except for some degenerate cases.

Find the equations of the conic sections resulting from the intersection of a double cone, \( z^2 = x^2 + y^2 \), with the following cutting planes:

a. \( z = 3x + 2 \).

b. \( z = \frac{1}{2}x + 3 \).

Solution: (a) \( x^2 + y^2 = (3x + 2)^2 = 9x^2 + 12x + 4 \) or \( 8x^2 - y^2 + 12x + 4 = 0 \); (b) \( x^2 + y^2 = \left(\frac{1}{2}x + 3\right)^2 = \frac{1}{4}x^2 + 3x + 9 \) or \( 3x^2 + 4y^2 - 12x - 36 = 0 \).

Conic sections in nature

Conic sections are important because they model important physical processes in nature. It can be shown that any body under the influence of an inverse square law force must have a trajectory of one of the conic sections. Heavenly bodies attract each other with a gravitational force that is inversely proportional to the square of the distance between them. This law, Newton's Law of Gravitation, is an example of an inverse square law. Hence, the trajectories of heavenly bodies are conic sections (circle, ellipses, parabolas, and hyperbolas.) Coulomb's Law of the attraction of charged particles is another example of an inverse square law. Therefore, the trajectories of charged particles (e.g., electrons) are conic sections. So, from the large scale of the universe to the microscopic scale of the atom, conic sections occur in nature.

Determining conic sections

What determines whether the intersection of the plane and the double cone results in a circle, a parabola, an ellipse, a hyperbola or one of the degenerate cases? Draw a side view of a double cone being intersected by a plane. The type of conic section depends on the angle \( \alpha \) that the cutting plane makes with the horizontal and how this angle compares with the angle \( \beta \) made by the cone with the horizontal.

1. If \( \alpha = 0 \), then the resulting conic section is a circle.
2. If \( \alpha = \beta \), then the resulting conic section is a parabola.
3. If \( \alpha < \beta \), then the resulting conic section is an ellipse.
4. If $\alpha > \beta$, then the resulting conic section is a hyperbola.

Find the angle between the cone, $z^2 = x^2 + y^2$, and the cutting plane, $z = 2x + 3$.

Classify the conic. Solution: $\alpha = \tan^{-1}(2) = 63.4^\circ > 45^\circ$, hyperbola.

**Degenerate conic sections**

If the cutting plane passes through the base of the cone, then we may also obtain a point, a line, or a pair of intersecting lines. These are known as *degenerate conic sections*.

- The degenerate form of a circle is a point.
- The degenerate form of a parabola is a line or two parallel lines.
- The degenerate form of an ellipse is a point.
- The degenerate form of a hyperbola consists of two intersecting lines.

Determine the type of degenerate conic section for $x^2 - 2y^2 + 2x + 8y - 7 = 0$.

$x^2 - 2y^2 + 2x + 8y - 7 = 0$ is equivalent to $\left(x^2 + 2x + 1\right) - 2\left(y^2 - 4y + 4\right) = 0$. The last equation is equivalent to $(x + 1)^2 - 2(y - 2)^2 = 0$ (a degenerate hyperbola), whose solution is the point $(-1, 2)$.

**Classification of conics by general quadratic equation**

Conics can be classified (or distinguished) by computing the discriminant, $B^2 - 4AC$, of the general quadratic equation, $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. We will not be studying any conics with the $Bxy$ term, so we will always assume that $B = 0$. If $B \neq 0$, then the conic is rotated such that its major (and minor) axis is no longer parallel to one of the coordinate axes.

1. If $B^2 - 4AC > 0$, then the conic is a hyperbola. This is equivalent to showing that $A$ and $C$ have opposite signs.
2. If $B^2 - 4AC = 0$, then the conic is a parabola. This is equivalent to showing that either $A$ or $C$ is equal to zero.
3. If $B^2 - 4AC < 0$, then the conic is either an ellipse or a circle. This is equivalent to showing that $A$ and $C$ have the same sign. Circles are distinguished from ellipses when $A = C$.

Identify each of the following conics:

- $x^2 + 2y^2 - 4x + 6y - 1 = 0$ (ellipse)
- $2x^2 + 2y^2 - 4x + 6y - 1 = 0$ (circle)
- $x^2 - 2y^2 - 4x + 6y - 1 = 0$ (hyperbola)
- $x^2 - 4x + 6y - 1 = 0$ (parabola)

**Standard form equations of conic sections**

In Algebra II conic sections were defined geometrically, i.e., as a set of points (called a *locus*) satisfying a distance relationship between two points (circle, ellipse, and hyperbola) and a point and a line (parabola). We can also define ellipses and hyperbolas by considering the locus of points from two points and a line (directrix.)

You should be familiar with using these equations to solve problems involving conic sections. For example, let the points $(-1, 6)$ and $(3, -2)$ be the endpoints of a diameter of a circle. Determine the y-intercepts of the circle.
Solution. The midpoint of the diameter is the center of the circle and is the point (1, 2). Since the radius is the distance from the center to any point on the circle, the radius is the distance between the points (1, 2) and (3, -2) or
\[
\sqrt{(3-1)^2 + (-2-2)^2} = \sqrt{4+16} = \sqrt{20}.
\]
Using the standard form of the circle, the equation of the circle is 
\[
(x - 1)^2 + (y - 2)^2 = 20.
\]
We can use this formula to find the y-intercepts by substituting \( x = 0 \) into the equation.
\[
(0 - 1)^2 + (y - 2)^2 = 20 \Rightarrow (y - 2)^2 = 19 \Rightarrow y = \pm \sqrt{19} + 2.
\]

Parabolas
We have seen parabolas before as the shape of the graph of a quadratic function. Here, we give a more general definition of a parabola that emphasizes its geometric properties.

A parabola is the set of all points in a plane whose distance from a fixed point is equal to its distance from a fixed line. The fixed point is called the focus and the fixed line is the directrix. The line passing through the focus and perpendicular to the directrix is called its axis of symmetry, and the point where the parabola intersects its axis of symmetry is called its vertex. The line segment that joins two points on the parabola, passes through the focus, and is perpendicular to the axis of symmetry is called the focal width. The vertex is midway between the focus and the directrix. The graph of the parabola always curves around the focus and away from the directrix. The focal width is useful when sketching graphs of parabolas.

We will use this definition to derive an equation of a parabola with vertex at the origin, focus at \((0, p)\), and directrix \(y = -p\). See figure 3a on p. 683. Using the distance formula between two points, we can write
\[
\sqrt{(x-0)^2 + (y-p)^2} = d_1.
\]
Since the distance \(d_2\) is just the vertical distance between P and Q, then \(d_2 = y + p\). By the definition of a parabola, we have \(d_1 = d_2\) or \(\sqrt{(x-0)^2 + (y-p)^2} = y + p\). Squaring both sides, we obtain
\[
(x-0)^2 + (y-p)^2 = (y + p)^2 \Rightarrow x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2.
\]
Combining like terms, we obtain \(x^2 = 4py\).

See figure 6 on p. 685. Depending on the relative location of the focus and the directrix, the parabola has different orientations and the equations differ too. If the \(x\)-term is squared, the parabola opens up or down. If the \(y\)-term is squared, the parabola opens left or right. If \(p\) is positive, the parabola opens up or to the right; if \(p\) is negative, the parabola opens down or to the left.

Translations
We can translate the graph of a parabola \(h\) units horizontally by replacing \(x\) with \((x - h)\) and \(k\) units vertically by replacing \(y\) with \((y - k)\). This leads to four standard equations of parabolas, one for each orientation.

Convert equations of parabolas to standard form, sketch graphs
Graph the parabola \(y^2 + 2y + 8x + 17 = 0\) and specify its vertex, focus, directrix, and axis of symmetry.
\[
y^2 + 2y + 8x + 17 = 0
\]
\[
y^2 + 2y = -8x - 17 \quad \text{(separate } x\text{- and } y\text{-terms)}
\]
\[
y^2 + 2y + 1 = -8x - 17 + 1 = -8x - 16 \quad \text{(complete the squared term)}
\]
\[(y + 1)^2 = -8(x + 2)\] (factor both sides)

Therefore, the vertex is \((-2, -1)\). Since the \(y\)-term is squared and \(p\) is negative, the graph of the parabola opens left. The focus is a distance \(p\) from the vertex. Since \(4p = 8\), \(p = 2\) and the focus is \((-4, -1)\). The directrix is a distance \(p\) from the vertex on the opposite side from the focus, i.e., \(x = 0\). The axis of symmetry is the horizontal line passing through the vertex \(y = -1\).

**Finding a parabola from its focus and directrix**

Given the focus \((3, 1)\) and the directrix \(y = -3\), find the equation of the parabola. To find the standard form, we need to find \(h, k,\) and \(p\); and we need to determine which of the four standard forms to use. The vertex lies halfway between the vertex and the focus or at the point \((3, -1)\). So \(h = 3\) and \(k = -1\). To find \(p\), we find the distance between the focus and the vertex, here 2. A quick sketch of the directrix and the vertex will determine which standard form to use. Since the directrix is horizontal and lies below the vertex, the parabola curves up; and the correct standard form is \((x - h)^2 = 4p(y - k)\). Therefore, we have \((x - 3)^2 = 4(2)(y + 1) = 8(y + 1)\).

Problem 11.R.74. See figure on p. 750. From the figure, the coordinates of \(A\) are \((r_1, 0)\) and of \(V\) are \(\left(\frac{r_1 + r_2}{2}, y_1\right)\). So, \(VO^2 = \left(\frac{r_1 + r_2}{2}\right)^2 + y_1^2\) and \(VA^2 = \left(\frac{r_1 + r_2}{2} - r_1\right)^2 + y_1^2\). After substituting, we get \(VO^2 - VA^2 = \left(\left(\frac{r_1 + r_2}{2}\right)^2 + y_1^2\right) - \left(\left(\frac{r_1 + r_2}{2} - r_1\right)^2 + y_1^2\right)\). The right side simplifies to \(r_1 r_2\).

**Ellipses**

An ellipse is the set of all points in the plane, the sum of whose distances from two fixed points (called the **foci**, plural of focus) is constant.

Using this definition, the text derives two standard forms for the equation of an ellipse using again the distance formula. The two forms depend on the orientation of the ellipse. See figures in property summaries on pp. 698 and 701.

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1
\]

Read definitions for **center**, **foci**, **major (focal) axis**, **minor axis**, **vertices** (major axis), **endpoints of minor axis**, **focal width**, and **eccentricity**. The letter \(a\) represents the distance from the center to a vertex, \(b\) represents the distance from the center to an endpoint on the minor axis, and \(c\) represents the distance between the center and a focus. These three values are related by the equation \(a^2 = b^2 + c^2\). Notice the length of the major axis is \(2a\) and the length of the minor axis is \(2b\). The center is located midway between the foci, midway between the vertices, and midway between the endpoints on the minor axis. The vertices lie along the major axis. The orientation of the ellipse depends on the parameter \(a\); the term with the larger denominator determines the major axis.

If we translate the center of an ellipse to the point \((h, k)\), then we obtain the standard forms:

\[
\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1 \quad \frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1
\]
The foci, vertices, and endpoints are translated similarly.

**Curve sketching**

Sketch the graph of \( 9x^2 + 25y^2 - 18x + 100y - 16 = 0 \). Identify the center, major axis, foci, vertices, endpoints of the minor axis, and x-intercepts.

1. First, we need to convert the equation to standard form.
   \[
   9x^2 + 25y^2 - 18x + 100y - 16 = 0 \Rightarrow 9(x^2 - 2x) + 25(y^2 + 4y) = 16 \Rightarrow \\
   9(x^2 - 2x + 1) + 25(y^2 + 4y + 4) = 16 + 9 + 100 \Rightarrow 9(x - 1)^2 + 25(y + 2)^2 = 225 \Rightarrow \\
   \frac{(x - 1)^2}{25} + \frac{(y + 2)^2}{9} = 1.
   \]

2. The center \((h, k)\) is \((1, -2)\).
3. The major axis is determined by the larger denominator; here, the major axis is parallel to the x-axis and passes through the center, i.e., \(y = -2\).
4. To determine the vertices and endpoints on the minor axis, we need to find the values of the parameters \(a\) and \(b\). \(a = \sqrt{25} = 5\) and \(b = \sqrt{9} = 3\). Therefore, the vertices are \((-4, -2)\) and \((6, -2)\). The endpoints of the minor axis are \((1, -5)\) and \((1, 1)\).
5. To determine the foci, we need to find the value of the parameter \(c\). Since \(c^2 = a^2 - b^2\), \(c = \sqrt{a^2 - b^2} = \sqrt{25 - 9} = \sqrt{16} = 4\). Therefore, the foci are \((-3, -2)\) and \((5, -2)\).
6. To determine the x-intercepts, substitute \(y = 0\) into the equation.
   \[
   \frac{(x - 1)^2}{25} + \frac{(0 + 2)^2}{9} = 1 \Rightarrow \frac{(x - 1)^2}{25} + \frac{4}{9} = 1 \Rightarrow \frac{(x - 1)^2}{25} = \frac{5}{9} \Rightarrow (x - 1)^2 = \frac{125}{9} \Rightarrow \\
   x - 1 = \pm \sqrt{\frac{125}{9}} \Rightarrow x = 1 \pm \sqrt{\frac{125}{9}} = 4.73, -2.73
   \]
7. Now we can draw the graph.

**Eccentricity**

The *eccentricity* of an ellipse or a hyperbola is defined to be the number \(e = \frac{c}{a}\).

The eccentricity is the ratio of the distance from the center to a focus divided by the distance from the center to a vertex. Since \(c < a\) for an ellipse, we know \(0 < e < 1\). If \(e\) is close to zero, the shape is more circular; if \(e\) is close to one, the shape is more elongated. Since \(c > a\) for a hyperbola, we know \(e > 1\). If \(e\) is close to one, the shape is more elongated; if \(e\) is close to \(\sqrt{2}\) one, the shape is more like an X; and if \(e \to \infty\), then the shape again is more elongated.

Sketch a few ellipses and label their foci. The shape of an ellipse depends on how close the foci are to the center. If the foci are close to the center, the shape is more circular. If the foci are closer to the vertices, the shape is more elongated. A number that measures the shape is the *eccentricity*, where \(e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}\). The eccentricity is the ratio of the distance from the center of the ellipse to a focus divided by the distance from the center to a vertex. Since \(c < a\), we know \(0 < e < 1\). If \(e\) is close to zero, the shape is more circular; if \(e\) is close to one, the shape is more elongated.
Sketch \( \frac{x^2}{36} + \frac{y^2}{25} = 1 \) and \( \frac{x^2}{36} + \frac{y^2}{1} = 1 \) and label their foci, \((\pm \sqrt{11},0)\) and \((\pm \sqrt{35},0)\). The eccentricity of the first graph is \( e = \frac{c}{a} = \frac{\sqrt{11}}{6} \approx 0.55 \), and the eccentricity of the second graph is \( e = \frac{c}{a} = \frac{\sqrt{35}}{6} \approx 0.99 \).

Since \( c > a \) for a hyperbola, we know \( e > 1 \). If \( e \) is close to one, the shape is more elongated; if \( e \) is close to \( \sqrt{2} \) one, the shape is more like an X; and if \( e \to \infty \), then the shape again is more elongated.

**Finding an ellipse from its graph**

Given the foci of an ellipse are (-3, 0) and (-1, 0) and the eccentricity is 1/3, find the equation of the ellipse and sketch its graph.

We need to find \( a, b, h \) and \( k \) and determine which standard form to use. Since the foci lie on the major axis, the major axis is horizontal and the standard form is \( \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \). Since the center is the midpoint of the foci, the center here is (-2, 1); therefore, \( h = -2 \) and \( k = 1 \). We need to find \( a \) and \( b \). Since \( c \) is the distance from the center to a foci, \( c = 1 \). Since \( e = \frac{c}{a} = \frac{1}{3} \), we know \( a = 3 \). Since \( b^2 = a^2 - c^2 \), \( b^2 = 8 \) and \( b = 2\sqrt{2} \). The equation of the ellipse is \( \frac{(x+2)^2}{9} + \frac{(y-1)^2}{8} = 1 \).

See figure for 11.R.70. The figure shows an ellipse and a parabola. The curves are symmetric about the x-axis and they both have an x-intercept of 5. Find the equations of the ellipse and the parabola, given that the point (3, 0) is a focus of both curves.

Solution. For the parabola, the vertex is (5, 0), so we know \( y^2 = 4p(x-5) \). Since the focus is (3, 0), then \( p = -2 \) and the equation of the parabola is \( y^2 = -8(x-5) \). For the ellipse, \( a = 5 \) and \( c = 3 \), so \( b^2 = a^2 - c^2 = 25 - 9 = 16 \), so \( b = 4 \). Thus the equation of the ellipse is \( \frac{x^2}{25} + \frac{y^2}{16} = 1 \).

**Hyperbolas**

A hyperbola is the set of all points in the plane, the difference of whose distances from two fixed points (called the foci) is a positive constant.

Using this definition, the text derives two standard forms for the equation of a hyperbola using again the distance formula. The two forms depend on the orientation of the hyperbola. See figures in property summaries on pp. 713 and 715.

\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{and} \quad \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1
\]

Read definitions of center, foci, focal axis, major (transverse) axis, minor (conjugate) axis, vertices, asymptotes, and eccentricity. The three values \( a, b, \) and \( c \) are related by a new equation \( c^2 = a^2 + b^2 \). This is because the foci lie inside the graph of the hyperbola and away from the center; therefore, \( c > a \). Unlike with the ellipse \( a \) is not always greater than \( b \). The length of the major axis is still \( 2a \) and the length of the minor
axis is still 2b. The orientation of the hyperbola depends on the parameter a; the term with the positive denominator determines the major axis.

It appears from the graphs that the graph of a hyperbola approaches one of two slant asymptotes. Let's see why. Using the second equation, let's solve for x.

\[
\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \implies \frac{y^2}{a^2} = \frac{x^2}{b^2} + 1 \implies \frac{y^2}{a^2} = \frac{x^2 + b^2}{b^2} \implies \frac{y^2}{a^2} = \frac{a^2(x^2 + b^2)}{b^2} \implies y = \pm \frac{a}{b} \sqrt{x^2 + b^2}.
\]

If we let x grow very large such that \(x \gg b\), then \(y = \pm \frac{a}{b} \sqrt{x^2 + b^2} \approx \pm \frac{a}{b} x\). The lines \(y = \frac{a}{b} x\) and \(y = -\frac{a}{b} x\) are the asymptotes of the hyperbola and help us draw its graph.

The asymptotes for the first form, \(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\), are \(y = \frac{b}{a} x\) and \(y = -\frac{b}{a} x\).

If we translate the center of a hyperbola to the point (h, k), then we obtain the standard forms:

\[
\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1 \quad \text{and} \quad \frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1
\]

The foci, vertices, and asymptotes are translated similarly.

**Curve sketching**

Sketch the graph of the hyperbola \(\frac{(y - 3)^2}{9} - \frac{(x - 4)^2}{25} = 1\). Identify the center, transverse axis, conjugate axis, foci, vertices, asymptotes, and x-intercepts.

1. The center (h, k) is (4, 3).
2. The major axis is determined by the positive rational term; here, the major axis is the vertical line parallel to the y-axis passing through the vertex, x = 4.
3. To determine the vertices and endpoints on the minor axis, we need to find the values of the parameters a and b. \(a = \sqrt{9} = 3\) and \(b = \sqrt{25} = 5\). Note that a is not greater than b. The vertices are (4, 0) and (4, 6). The endpoints of the minor axis are (-1, 3) and (9, 3).
4. To determine the foci, we need to find the value of the parameter c. Since \(c^2 = a^2 + b^2\), \(c = \sqrt{a^2 + b^2} = \sqrt{25 + 9} = \sqrt{34} \approx 5.8\). Therefore, the foci are \((4, 3 - \sqrt{34})\) and \((4, 3 + \sqrt{34})\) or approximately \((4, -2.8)\) and \((4, 8.8)\).
5. The asymptotes are \(y = \pm \frac{a}{b} (x - h) + k\), i.e., \(y = \pm \frac{3}{5} (x - 4) + 3\).
6. To determine the x-intercepts, substitute y = 0 into the equation.

\[
\frac{(x - 1)^2}{25} + \frac{(0 + 2)^2}{9} = 1 \implies \frac{(0 - 3)^2}{9} - \frac{(x - 4)^2}{25} = 1 \implies \frac{9}{9} - \frac{(x - 4)^2}{25} = 1 \implies \frac{(x - 4)^2}{25} = 0 \implies x = 4.
\]

Therefore, we have only one x-intercept, (4, 0).

7. Now we can draw the graph.

**Finding a hyperbola from its graph**

Given the asymptotes of a hyperbola are \(y = \pm (1/2)x\) and the vertices are \((\pm 2, 0)\) find the equation of the hyperbola.
We must solve for \(a, b, h,\) and \(k\). From the equations of the asymptotes, we know the center of the hyperbola is the origin, i.e., \((h, k)\) is \((0, 0)\). Since the vertices lie along the \(x\)-axis, we know the orientation and the general form, \(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\). Since the vertices are a distance of 2 from the center, we know \(a = 2\). We also know the form of the asymptotes are \(y = \pm \frac{b}{a} x\), and, therefore, \(b = 1\). Substituting for \(a, b, h,\) and \(k,\) into the general form, we obtain \(\frac{x^2}{4} - \frac{y^2}{1} = 1\).

**Focal width**

The *focal chord* of a parabola, an ellipse, or a hyperbola is a chord passing through the focus. For graphing purposes, it is useful to know the length of the focal chord perpendicular to the (major) axis. This length is called the *focal width.* See figure 8 on p. 685. The focal width for this parabola is equal to \(4p\). This is true generally for all parabolas. The focal widths of ellipses and hyperbolas can also be computed. Knowing the length of the focal width is useful for graphing conic sections.

Find the focal width of the parabola, \(x^2 = 4py, \ p > 0\). Since the focus is the point \((0, p)\) and the axis is the vertical line \(x = 0\), the focal width is the length of the focal chord along the perpendicular line \(y = p\). Substituting \(p\) for \(y\) into the equation, we get \(x^2 = 4p^2\). Solving for \(x\), we get \(x = \pm 2p\) and the endpoints of the focal chord are \((-2p, p)\) and \((2p, p)\). The distance between these points is \(4p\). This is true for all parabolas.

Find the focal width of the parabola, \((y + 1)^2 = -8(x + 2)\). The endpoints of the chord passing through the focus and perpendicular to the axis of symmetry are \((-4, y)\). Let’s solve for \(y\). \((y + 1)^2 = -8(-4 + 2) = -8(-2) = 16 \Rightarrow y + 1 = \pm 4\). Solving these two equations, we obtain \(y = 3, -5\) and the two points are \((-4, 3)\) and \((-4, -5)\). The distance between these two points is \(3 - (-5) = 8\).

Find the focal width of the ellipse, \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\), where \(a > b\). Since the focal width passes through the focus \((c, 0)\), the points on the graph of the ellipse that lie on the focal chord are \((c, \pm y_1)\). We need to solve for \(y_1\) to determine the focal width, \(2y_1\). For ellipses, we have \(a^2 = b^2 + c^2\), so \(c = \sqrt{a^2 - b^2}\). So, \(\frac{\sqrt{a^2 - b^2}}{a} + \frac{y_1^2}{b^2} = 1\). Solving for \(y_1\) we get \(y_1 = \pm \frac{b^2}{a}\). Therefore, the focal width is \(\frac{2b^2}{a}\).

Find the focal width of the hyperbola, \(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\). Since the focal width passes through the focus \((c,0)\), the points on the graph of the ellipse that lie on the focal chord are \((c, \pm y_1)\). We need to solve for \(y_1\) to determine the focal width, \(2y_1\). For hyperbolas,
we have \( c^2 = a^2 + b^2 \), so \( c = \sqrt{a^2 + b^2} \). So, \( \frac{\left(\sqrt{a^2 + b^2}\right)^2}{a^2} - \frac{y_1^2}{b^2} = 1 \). Solving for \( y_1 \) we get \( y_1 = \pm \frac{b^2}{a} \). Therefore, the focal width is \( \frac{2b^2}{a} \), exactly the same focal width as for the ellipse.

**Applications**

You should be familiar with using these equations to solve problems involving conic sections. For example, let the points (-1, 6) and (3, -2) be the endpoints of a diameter of a circle. Determine the \( y \)-intercepts of the circle.

Solution. The midpoint of the diameter is the center of the circle and is the point (1, 2). Since the radius is the distance from the center to any point on the circle, the radius is the distance between the points (1, 2) and (3, -2) or
\[
\sqrt{(3-1)^2 + (-2-2)^2} = \sqrt{4 + 16} = \sqrt{20}.
\]
Using the standard form of the circle, the equation of the circle is \((x - 1)^2 + (y - 2)^2 = 20\). We can use this formula to find the \( y \)-intercepts by substituting \( x = 0 \) into the equation.
\[
(0-1)^2 + (y-2)^2 = 20 \Rightarrow (y-2)^2 = 19 \Rightarrow y = \pm \sqrt{19} + 2.
\]
See figure for 11.R.70. The figure shows an ellipse and a parabola. The curves are symmetric about the \( x \)-axis and they both have an \( x \)-intercept of 5. Find the equations of the ellipse and the parabola, given that the point (3, 0) is a focus of both curves.

Solution. For the parabola, the vertex is (5, 0), so we know \( y^2 = 4p(x - 5) \). Since the focus is (3, 0), then \( p = -2 \) and the equation of the parabola is \( y^2 = -8(x - 5) \). For the ellipse, \( a = 5 \) and \( c = 3 \), so \( b^2 = a^2 - c^2 = 25 - 9 = 16 \), so \( b = 4 \). Thus the equation of the ellipse is \( \frac{x^2}{25} + \frac{y^2}{16} = 1 \).