MATH 102 - APPLIED LINEAR ALGEBRA
FINAL EXAM

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TA: 

YOU MAY ASSUME THE RESULTS OF PREVIOUS PARTS OF A PROBLEM TO PROVE THE PART OF
THE PROBLEM YOU ARE WORKING ON.

ANSWERS TO THE TRUE/FALSE QUESTIONS
DO NOT NEED TO BE JUSTIFIED; HOWEVER,
INCORRECT ANSWERS TO THE TRUE/FALSE
QUESTIONS ARE PENALIZED. IN PARTICULAR,
A CORRECT ANSWER IS WORTH 5 POINTS, AN
INCORRECT ANSWER IS WORTH -5 POINTS,
AND A BLANK ANSWER IS WORTH 0 POINTS.

REMEMBER THIS EXAM IS GRADED BY A
HUMAN BEING. WRITE YOUR SOLUTIONS
NEATLY AND COHERENTLY, OR THEY RISK
NOT RECEIVING FULL CREDIT.

THIS EXAM WILL BE SCANNED. MAKE SURE
YOU WRITE ALL SOLUTIONS IN THE SPACES
PROVIDED. DO NOT PHYSICALLY REMOVE
ANY OF THE PAGES.

THE EXAM CONSISTS OF 14 TRUE/FALSE QUES-
TIONS AND 7 LONGER FORMAT QUESTIONS.
YOUR ANSWERS TO THE LONGER FORMAT
QUESTIONS SHOULD BE CAREFULLY JUSTI-
FIED. YOU ARE ALLOWED TO USE RESULTS
FROM THE TEXTBOOK, HOMEWORK, AND
LECTURE, BUT THEY SHOULD BE CLEARLY
REFERENCED (FOR EXAMPLE, “BY THE SPEC-
TRAL THEOREM, ...”).
1. (70 points) Label the following statements as true or false. Any ambiguous answer (for example, resembling a hybrid of T and F) will be treated as an incorrect answer.

(a) False If $S$ is a linearly dependent set, then each vector in $S$ is a linear combination of the other vectors in $S$.

Not every vector has to be a linear combination of the others, just one.

(b) True If $A \in M_{n \times n}(\mathbb{C})$, then $\det(A^*) = \overline{\det(A)}$.

$$A^* = (\bar{A})^+ \quad \text{and} \quad \det(A) = \overline{\det(A^*)}$$

(c) True Let $V$ and $W$ be vector spaces over a common field $F$, and let $\{v_1, \ldots, v_n\}$ be a basis for $V$. Suppose that we are give $n$ vectors $w_1, \ldots, w_n \in W$ (not necessarily distinct, so it could be that $w_i = w_j$ for $i \neq j$). Then there exists a unique linear transformation $T : V \to W$ such that $T(v_i) = w_i$ for each $i = 1, \ldots, n$.

Simply define $T(v_i) = w_i$ and extend by linearity

(d) False If $T : V \to V$ is a linear operator on a finite-dimensional inner product space $(V, \langle \cdot, \cdot \rangle)$ with an ordered orthonormal basis $\beta = (v_1, \ldots, v_n)$, then

$$[T]_\beta(i, j) = \langle T(v_i), v_j \rangle,$$

where $[T]_\beta(i, j)$ denotes the $(i, j)$-th entry of the matrix $[T]_\beta$.

$$\begin{bmatrix} T \end{bmatrix}_\beta (i, j) = \langle T(v_j), v_i \rangle$$

(e) True If $A, B \in M_{n \times n}(F)$, then

$$\text{tr}(AB) = \text{tr}(BA).$$

$$\text{tr} (AB) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} B_{ji} = \text{tr}(BA)$$
(f) **True**  If $O$ is an orthogonal matrix, then $\det(O) \in \{-1, 1\}$.

\[
1 = \det(I_n) = \det(00^T) = \det(O) \det(O^T) = \det(O)^2
\]

and $\det(O) \in \mathbb{R}$ so $\det(O) \in \mathbb{R} -1, 1$

(g) **True**  Suppose that $T : V \to V$ is a linear operator on a finite-dimensional vector space $V$. If $\text{rank}(T) < \dim(V)$, then $T$ has an eigenvector.

$\dim(\ker(T)) \geq 1$ by dimension theorem, any non-zero vector in $\ker(T)$ will be an eigenvector.

(h) **True**  Let $T : V \to V$ be a linear operator. If $W, U, Z$ are each $T$-invariant subspaces of $V$, then so is $(W \cap U) + Z$.

Suppose $V = a + b$ where $a \in W \cap U$ and $b \in Z$

then $T(a) \in W \cap U$ so $T(v) = T(a) + T(b) \in (W \cap U) + Z$

(T(b) \in Z)

(i) **True**  Suppose that $V$ is an inner product space with subspaces $W, U \subseteq V$.

If $W \subseteq U$, then $U^\perp \subseteq W^\perp$.

If $v \in U^\perp$ then $\langle v, u \rangle = 0$ $\forall u \in U$

but $W \subseteq U$ so $\langle v, u \rangle = 0$ $\forall u \in W$

meaning $v \in W^\perp$

(j) **True**  If $A \in M_{n \times n}(\mathbb{C})$ is a unitary matrix, then the transpose $A^T$ is also a unitary matrix.

\[
A^T A^{T^*} = (A^*) (A^*)^* = (A^*) (A^{T^*}) = (A^*) (A) = (A^{T^*} A) = (I_n) = I_n
\]
(k) **False** Suppose that $T : V \rightarrow V$ is a linear operator on a finite-dimensional inner product space $V$ with an orthonormal basis $\beta = \{v_1, \ldots, v_n\}$. If $\|Tv_i\| = \|v_i\|$ for every $v_i \in \beta$, then $\|Tv\| = \|v\|$ for every $v \in V$.

Consider $L_A$ where $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

(l) **True** Suppose that $U, A \in M_{n \times n}(\mathbb{C})$. If $U$ is a unitary matrix and $A$ is unitarily equivalent to $U$, then $A$ is a unitary matrix.

$$A = VUV^* \quad \text{for some unitary matrix } V$$

so $A$ is unitary

(m) **True** If $P : V \rightarrow V$ and $Q : V \rightarrow V$ are orthogonal projections on an inner product space $V$ and $PQ = T_0$ (the zero operator), then $QP = T_0$.

$$(PQ)^* = T_0^* = T_0$$

$$Q^*P^* = QP$$

(n) **False** If $P : V \rightarrow V$ is an orthogonal projection, then $I_V + P$ is an orthogonal projection (here, $I_V : V \rightarrow V$ denotes the identity operator).

Let $P = I_V$. 
2. (40 points) Let $V$ be a vector space with subspaces $W_1$ and $W_2$. We do not make any assumptions about the dimensions of our vector space/subspaces for this problem.

(a) (20 pts) Prove that $V = W_1 \oplus W_2$ if and only if each vector $v \in V$ can be uniquely written as $v = x_1 + x_2$, where $x_1 \in W_1$ and $x_2 \in W_2$.

$(\Rightarrow)$: Suppose $V = W_1 \oplus W_2$, i.e., $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$, then $V = W_1 + W_2$ already tells us that each vector $v \in V$ can be written as a sum $v = x_1 + x_2$ for some $x_1 \in W_1$ and $x_2 \in W_2$.

Assume $v = x_1 + x_2 = y_1 + y_2$, where also $y_1 \in W_1$ and $y_2 \in W_2$.

Then $x_1 - y_1 = y_2 - x_2$ is in both $W_1$ and $W_2$ so $x_1 - y_1 = y_2 - x_2 = 0$ since $W_1 \cap W_2 = \{0\}$, meaning $x_1 = y_1$ and $x_2 = y_2$.

$(\Leftarrow)$: The stated property implies $V = W_1 + W_2$, so we just need to show $W_1 \cap W_2 = \{0\}$.

Let $w \in W_1 \cap W_2$. Then $w = w + 0 = 0 + w$ where $w \in W_1$ and $0 \in W_2$, and $0 \in W_1$ and $w \in W_2$. The uniqueness property then tells us that $w = 0$. 
(b) (20 pts) Suppose that $V = W_1 \oplus W_2$. Recall that a linear operator $P : V \to V$ is the projection on $W_1$ along $W_2$ if, whenever $x = x_1 + x_2$, with $x_1 \in W_1$ and $x_2 \in W_2$, we have $P(x) = x_1$.

Prove that if $T : V \to V$ satisfies $T^2 = T$, then $V = R(T) \oplus N(T)$ and $T$ is the projection on $R(T)$ along $N(T)$.

**Claim** \[ R(T) + N(T) = V \]

**Proof** \[
 v = T(v) + (v - T(v))
\]

Clearly, $T(v) \in R(T)$

Note that $T(v - T(v)) = T(v) - T^2(v) = T(v) - T(v) = 0$

So $v - T(v) \in N(T)$ and we are done.

**Claim** \[ R(T) \cap N(T) = \{0\} \]

Let $y \in R(T) \cap N(T)$

Then $y = T(x)$ for some $x \in V$ since $y \in R(T)$

and $T^2(x) = T(y) = 0$ since $y \in N(T)$

but $T^2(x) = T(x) = y$ so $y = 0$

Finally, for $v \in V$, we can write $v = v_1 + v_2$

where $v_1 \in R(T)$ and $v_2 \in N(T)$

Then \[
 T(v) = T(v_1) + T(v_2) = T(v_1) + 0 = v_1
\]

since $T^2 = T$ and $v_1 \in R(T)$ and $v_2 \in N(T)$

This proves that $T$ is indeed the projection on $R(T)$ along $N(T)$.
3. (40 points) Let $V$ be an inner product space with inner product $\langle \cdot , \cdot \rangle$. We do not make any assumptions about the dimension of $V$ for this problem.

(a) (10 pts) Suppose that $x$ and $y$ are orthogonal vectors in $V$. Prove that

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$ 

\[
\|x + y\|^2 = \langle x + y, x + y \rangle \\
= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\
= \|x\|^2 + 0 + 0 + \|y\|^2
\]

(b) (5 pts) Assume that $P : V \to V$ is a projection (necessarily on $R(P)$ along $N(P)$). Define what it means for $P : V \to V$ to further be an orthogonal projection in terms of $R(P)$ and $N(P)$.

\[
R(P) \perp = N(P) \\
N(P) \perp = R(P)
\]
(c) (10 pts) Suppose that $P : V \to V$ is an orthogonal projection. Prove that for every $v \in V$,
\[ \|P(v)\| \leq \|v\|. \]

Let $V = V_1 + V_2$, where $V_1 \in \mathbb{R}(P)$ and $V_2 \in \mathbb{N}(P)$, then $P(v) = V_1$.

\[ \|v\|^2 = \|v_1 + v_2\|^2 = \|v_1\|^2 + \|v_2\|^2 \geq \|v_1\|^2 \]

\[ \text{part (a)} \]

So $\|v\| \geq \|v_1\| = \|P(v)\|$
(d) (15 pts) Assume that $P : V \to V$ is an orthogonal projection. For vectors $v \in V$ and $w \in R(P)$, prove that

$$\| v - w \|^2 \geq \| v - P(v) \|^2.$$ 

Hint: since $w \in R(P)$, you know that $w = P(w)$.

\[
\| v - w \|^2 = \| v - P(v) + P(v) - P(w) \|^2 \\
= \| (v - P(v)) + (P(v) - P(w)) \|^2 \\
= \| v - P(v) \|^2 + \| P(v) - P(w) \|^2 \geq \| v - P(v) \|^2 \\
\text{Since } v - P(v) \in N(CP) \text{ and } P(v - w) \in R(CP)
\]
4. (30 pts) Let $A$ be an $n \times n$ real symmetric or complex normal matrix.
(a) (10 pts) Prove that

$$
\text{tr}(A) = \sum_{i=1}^{n} \lambda_i \quad \text{and} \quad \text{tr}(A^*A) = \sum_{i=1}^{n} |\lambda_i|^2,
$$

where the $\lambda_i$'s are the (not necessarily distinct) eigenvalues of $A$.

We know that $A = U D U^*$ for $U$ an orthogonal matrix (if $A$ is real symmetric) or unitary (if $A$ is complex normal) where $D$ is the diagonal matrix of eigenvalues of $A$.

So,

$$
\text{tr}(A) = \text{tr}(UDU^*) = \text{tr}(U^*UD)
$$

$$
= \text{tr}(I_nD) = \text{tr}(D) = \sum_{i=1}^{n} \lambda_i;
$$

\[ \blacklozenge \]

$$
\text{tr}(A^*A) = \text{tr}((U^*DU^*)(UDU^*))
$$

$$
= \text{tr}(UD^*DU^*) = \text{tr}(U^*UD^*D)
$$

$$
= \text{tr}(D^*D)
$$

$$
= \sum_{i=1}^{n} \lambda_i \lambda_i
$$

$$
= \sum_{i=1}^{n} |\lambda_i|^2
$$

\[ \blacklozenge \]
(b) (10 pts) Prove that

$$\det(A) = \prod_{i=1}^{n} \lambda_i,$$

where the $\lambda_i$'s are the (not necessarily distinct) eigenvalues of $A$.

$$\det(A) = \det(UDU^*) = \det(U) \det(D) \det(U^*)$$

$$= \det(U) \det(U^*) \det(D)$$

$$= \det(UU^*) \det(D)$$

$$= \det(I_n) \det(D)$$

$$= \det(D) = \prod_{i=1}^{n} \lambda_i;$$
(c) (10 pts) Suppose that $U \in M_{n \times n}(\mathbb{C})$ is both unitary and self-adjoint with $\det(U) = -1$.

Prove that $-1$ is an eigenvalue of $U$.

U is unitary and self-adjoint so it is complex normal.

Part (b) says $\det(U) = \prod_{i=1}^{n} \lambda_i$,

where the $\lambda_i$ are the eigenvalues of $U$.

Since $U$ is unitary, $|\lambda_i| = 1$ for each $i$.

Since $U$ is self-adjoint, $\lambda_i \in \mathbb{R}$ for each $i$.

Hence $\lambda_i \in \{-1, 1\}$ for each $i$.

Since $\prod_{i=1}^{n} \lambda_i = -1$, at least one of the eigenvalues must be $-1$. 
5. (40 points) Let $A \in M_{n \times n}(\mathbb{C})$.
   (a) (5 pts) Define what it means for $A$ to be a unitary matrix.

   \[
   A^* A = A A^* = I_n
   \]

   (b) (15 pts) Prove that the columns of a unitary matrix $A$ are orthonormal.

   One can prove this directly using the equality $A^* A = I_n$

   Alternatively, the operator $L_A : \mathbb{C}^n \to \mathbb{C}^n$ is unitary since
   \[
   L_A L_A^* = L_A A^* = L I_n = I_{\mathbb{C}^n}
   \]
   \[
   L_A^* L_A = L A^* A = L I_n = I_{\mathbb{C}^n}
   \]

   Unitary operators take orthonormal bases to orthonormal bases.

   Consider the standard ordered basis $(e_1, \ldots, e_n)$ then
   \[
   (L_A(e_1), \ldots, L_A(e_n)) = (A e_1, \ldots, A e_n)
   \]
   is an orthonormal basis.

   But $A e_j$ is just the $j$th column of $A$. 
(c) (20 pts) Recall that a matrix \( A \in M_{n \times n}(\mathbb{C}) \) is called lower triangular if

\[ A(i, j) = 0 \quad \text{whenever} \quad i < j. \]

Prove that if a matrix \( A \in M_{n \times n}(\mathbb{C}) \) is both unitary and lower triangular, then it must be a diagonal matrix. If \( A(i, i) \) is a diagonal entry of \( A \), then what is the absolute value/modulus \( |A(i, i)| \)?

Let \( c_i \) be the \( i \)-th column of \( A \), where \( i = 1, \ldots, n \). We will prove that \( c_i \) does not have any nonzero entries below the \( i \)-th row by induction.

First, note that the eigenvalues of \( A \) are all of modulus 1 since \( A \) is unitary.

Moreover, since \( A \) is lower triangular, the diagonal entries are the eigenvalues, so \( |A(i, i)| = 1 \) for \( i = 1, \ldots, n \).

We will do our induction starting with the last column \( c_n \).

Clearly \( c_n \) has no nonzero entries below the \( n \)-th row because it only has \( n \) rows.

Assume the result is true for \( c_n \) down to \( c_m \) where \( m \leq n \) and consider column \( c_{m-1} \).

By part (b), \( \langle c_{m-1}, c_j \rangle = 0 \) for \( j = m, m+1, \ldots, n \).

However, since \( A \) is lower triangular and \( c_j \) has no nonzero entries below the \( j \)-th row,

\[ \langle c_{m-1}, c_j \rangle = \sum_{k=1}^{m-1} A(k, m-1)A(k, j) = 0 \]

since \( |A(j, j)| = 1 \), this means \( A(j, m-1) = 0 \) for \( j = m, \ldots, n \).

This completes our induction and shows that \( A \) is diagonal.
6. (40 points) Let \( V \) be a finite-dimensional inner product space with inner product \( \langle \cdot, \cdot \rangle \).

(a) (5 pts) State the Cauchy-Schwarz inequality.

\[
|\langle x, y \rangle|^2 \leq \langle x,x \rangle \langle y,y \rangle \quad \forall x, y \in V
\]

(b) (15 pts) A matrix \( A \in M_{n \times n}(\mathbb{C}) \) is said to be **positive definite** if \( A = A^* \) and

\[
\langle Ax, x \rangle_{\mathbb{C}^n} > 0 \quad \text{for each non-zero vector } x \in \mathbb{C}^n,
\]

where \( \langle \cdot, \cdot \rangle_{\mathbb{C}^n} \) is the standard inner product on \( \mathbb{C}^n \). Prove that the function

\[
\langle \cdot, \cdot \rangle_A : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}
\]

given by

\[
(x, y)_A = \langle Ax, y \rangle_{\mathbb{C}^n}
\]

defines an inner product on \( \mathbb{C}^n \).

1. \[
\langle x + z, y \rangle_A = \langle A(x + z), y \rangle_{\mathbb{C}^n} = \langle Ax + Az, y \rangle_{\mathbb{C}^n}
\] \[
= \langle Ax, y \rangle_{\mathbb{C}^n} + \langle Az, y \rangle_{\mathbb{C}^n}
\] \[
= \langle x, y \rangle_A + \langle z, y \rangle_A
\]

2. \[
\langle cx, y \rangle_A = \langle A(cx), y \rangle_{\mathbb{C}^n} = c \langle A(x), y \rangle_{\mathbb{C}^n}
\] \[
= c \langle x, y \rangle_A
\]

3. \[
\langle y, x \rangle_A = \langle Ay, x \rangle_{\mathbb{C}^n} = \langle y, Ax \rangle_{\mathbb{C}^n} = \overline{\langle Ax, y \rangle_{\mathbb{C}^n}}
\]

A self-adj. \[ \quad \overline{\langle x, y \rangle_A} \]

4. Let \( x \neq 0 \), then

\[
\langle x|x \rangle_A = \langle Ax, x \rangle_{\mathbb{C}^n} > 0 \quad \text{by assumption.}
\]
(c) (20 pts) Suppose that $A \in M_{n \times n}(\mathbb{C})$ is positive definite. Prove that

$$|A(i, j)|^2 \leq A(i, i)A(j, j),$$

for any $i, j \in \{1, \ldots, n\}$.

$$|A(i, j)|^2 = |\langle Ae_j, e_i \rangle|^2 = |\langle e_j, e_i \rangle_A|^2$$

$$\leq \langle e_j, e_j \rangle_A \langle e_i, e_i \rangle_A$$

$$= \langle Ae_j, e_j \rangle \langle Ae_i, e_i \rangle$$

$$= A(j, j) A(i, i)$$
7. (40 points) Let \((V, \langle \cdot, \cdot \rangle)\) be a finite-dimensional complex inner product space. Fix a subspace \(W \subseteq V\). Since \(V\) is finite-dimensional, we then know that \(V = W \oplus W^\perp\).

(a) (20 pts) Define a linear operator \(U : V \to V\) by \(U(v) = U(v_1 + v_2) = v_1 - v_2\), where \(v = v_1 + v_2\) is the unique decomposition of \(v\) as a sum of vectors \(v_1 \in W\) and \(v_2 \in W^\perp\) (since \(V = W \oplus W^\perp\)). Prove that \(U\) is self-adjoint and unitary.

\[
\begin{align*}
V &= V_1 + V_2 & \text{where } V_1 \subseteq W, \quad V_2 \subseteq W^\perp \\
W_1 &= W_1 + W_2 & \text{where } W_1 \subseteq W, \quad W_2 \subseteq W^\perp
\end{align*}
\]

Then

\[
\langle Uv, w \rangle = \langle v_1 - v_2, w_1 + w_2 \rangle = \langle v_1, w_1 \rangle + \langle v_1, w_2 \rangle - \langle v_2, w_1 \rangle - \langle v_2, w_2 \rangle
\]

and

\[
\langle v, Uw \rangle = \langle v_1 + v_2, w_1 - w_2 \rangle = \langle v_1, w_1 \rangle - \langle v_1, w_2 \rangle + \langle v_2, w_1 \rangle - \langle v_2, w_2 \rangle
\]

so

\[
\langle Uv, w \rangle = \langle v, Uw \rangle \quad \text{for every } v, w \in V
\]

meaning \(U\) is self-adjoint.

Observe that \(U^2 = I_V\) \((U^2(v) = U(U(v_1 + v_2)) = U(v_1 - v_2) = v_1 - (-v_2) = v_1 + v_2 = V)\)

and \(U^* = U\) since self-adjoint.

So, \(I_V = U^2 = U^*U = UU^*\) meaning \(U\) is unitary.
(b) (10 pts) Determine the eigenvalues of $U$ and the corresponding eigenspaces.

Note then $U(v_1) = v_1$ for every $v_1 \in W$

$U(v_2) = -v_2$ for every $v_2 \in W^\perp$

Since $V = W \oplus W^\perp$, we conclude

that the eigenvalues of $U$ are 1 and $-1$

with $E_1 = W$

$E_{-1} = W^\perp$
(c) (10 pts) For each of the eigenspaces $E_{\lambda_i}$ in part (b), let $P_i : V \to V$ be the orthogonal projection onto $E_{\lambda_i}$. Find a polynomial $q_i(t) \in \mathbb{C}[t]$ such that $q_i(U) = P_i$. You will need to do this for each eigenspace $E_{\lambda_i}$ where $i = 1, \ldots, k$ and $k$ is the number of distinct eigenvalues of $U$.

Hint: don’t try to be too clever.

Let $v \in V$ with $v = v_1 + v_2$ where $v_1 \in W$ and $v_2 \in W^\perp$.

Since $U(v_1 + v_2) = v_1 - v_2$,

we see that $(U + I_V)(v_1 + v_2) = v_1 - v_2 + v_1 + v_2 = 2v_1$,

so \[ \frac{U + I_V}{2}(v_1 + v_2) = v_1 = p_1(v) \]

So, $p_1 = \frac{U + I_V}{2}$.

Similarly, $(U - I_V)(v_1 + v_2) = v_1 - v_2 - v_1 - v_2 = -2v_2$.

so \[ p_2 = \frac{U - I_V}{2} = q_2(U) \] where \[ q_2(t) = \frac{t - 1}{-2} \]
8. (20 points) Let $T : V \to V$ be a normal linear operator on a finite-dimensional complex inner product space. Suppose that $p(t) \in \mathbb{C}[t]$ is a polynomial and that $\zeta$ is an eigenvalue of $p(T)$. Prove that there exists an eigenvalue $\lambda$ of $T$ such that $p(\lambda) = \zeta$.

By the spectral theorem,

$$T = \sum_{i=1}^{k} \lambda_i \mathbf{p}_i$$

where the $\lambda_i$ are the distinct eigenvalues of $T$ and $\mathbf{p}_i$ is the orthogonal projection onto the eigenspace $E_{\lambda_i}$.

Then

$$p(T) = \sum_{i=1}^{k} p(\lambda_i) \mathbf{p}_i$$

has the eigenvalues $p(\lambda_1), \ldots, p(\lambda_k)$.

Since $\zeta$ is an eigenvalue of $p(T)$, this means $\zeta = p(\lambda_i)$ for some $i \in \{1, \ldots, k\}$. 
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