MATH 10C - FINAL

Name (Last, First): ________________________________

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REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT.

THIS EXAM WILL BE SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS ON THE PAPER PROVIDED. DO NOT REMOVE ANY OF THE PAGES.

THE EXAM CONSISTS OF 9 QUESTIONS. YOUR ANSWERS SHOULD BE CAREFULLY JUSTIFIED.
1. (10 points) Find the area of the triangle in $\mathbb{R}^3$ with vertices $P = (1,0,0)$, $Q = (0,1,0)$, and $R = (0,0,2)$.

Proof. We need to compute $\|\overrightarrow{PQ} \times \overrightarrow{PR}\|$. Using any of the formulas we have for the cross product, we see that

$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -1, 1, 0 \rangle \times \langle -1, 0, 2 \rangle = \langle 2, 2, 1 \rangle$.

So, our answer is

$$\frac{\sqrt{2^2 + 2^2 + 1^2}}{2} = \frac{3}{2}.$$
2. (10 points) Are the planes \( x - 2y + z = 5 \) and \(-2(x - 1) + 4(y - 2) - (z - 3) = 2 \) parallel? If so, find the distance between them. If not, find the angle \( \theta \in [0, \pi] \) of intersection between them.

Proof. A normal vector for the first plane is \( \vec{n}_1 = \langle 1, -2, 1 \rangle \), while a normal vector for the second plane is \( \vec{n}_2 = \langle -2, 4, -1 \rangle \). Since these vectors are not non-trivial scalar multiples of each other, the two planes are not parallel. Our usual formula says that the angle of intersection \( \theta \) satisfies

\[
\cos(\theta) = \frac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\| \|\vec{n}_2\|}
\]

and so

\[
\theta = \cos^{-1}\left(\frac{1(-2) + (-2)(4) + (1)(-1)}{\sqrt{1^2 + (-2)^2 + 1^2} \sqrt{(-2)^2 + 4^2 + (-1)^2}}\right).
\]

One could also use the absolute value formula instead to get an acute angle \( \theta \in [0, \frac{\pi}{2}] \):

\[
\theta = \cos^{-1}\left(\frac{|1(-2) + (-2)(4) + (1)(-1)|}{\sqrt{1^2 + (-2)^2 + 1^2} \sqrt{(-2)^2 + 4^2 + (-1)^2}}\right).
\]
3. (15 points) Let \( f(x, y, z) = (x + y^2)e^{-z} \).

(a) (10 points) Find the direction of maximal decrease for \( f \) at the point \((1, 1, 1)\).

Proof. First, we compute the gradient:
\[
\nabla f(x, y, z) = \langle e^{-z}, 2ye^{-z}, -(x + y^2)e^{-z} \rangle.
\]

By the properties of the gradient, we then know that
\[ -\nabla f(1, 1, 1) = \langle -e^{-1}, 2e^{-1}, -2e^{-1} \rangle = \langle -e^{-1}, -2e^{-1}, 2e^{-1} \rangle \]
points in the direction of maximal decrease; however, the problem asks for direction. So, we convert this to a unit vector
\[
\frac{-\nabla f(1, 1, 1)}{\| -\nabla f(1, 1, 1) \|} = \frac{\langle -e^{-1}, -2e^{-1}, 2e^{-1} \rangle}{\sqrt{(-e^{-1})^2 + (-2e^{-1})^2 + (2e^{-1})^2}}.
\]
\[ \square \]
(b) (5 points) Find the directional derivative of $f$ at the point $(1, 1, 1)$ in the direction of the vector $\vec{v} = \langle 1, 0, 1 \rangle$.

Proof. The formula for the directional derivative is simply
\[
D_{\vec{u}} f(1, 1, 1) = \nabla f(1, 1, 1) \cdot \vec{u}
\]
\[
= \langle e^{-1}, 2e^{-1}, -2e^{-1} \rangle \cdot \frac{\langle 1, 0, 1 \rangle}{\sqrt{1^2 + 0^2 + 1^2}}
\]
\[
= \frac{(e^{-1})(1) + (-2e^{-1})(1)}{\sqrt{1^2 + 0^2 + 1^2}}.
\]
\[\square\]
4. (10 points) Find the tangent plane to the surface \(2yz = \ln(x + e^z)\) at \((x, y, z) = (0, 0, 0)\).

Proof. Let \(F(x, y, z) = \ln(x + e^z) - 2yz\) (you could also consider \(-F(x, y, z)\)). Then the problem is asking for the tangent plane to the surface \(F(x, y, z) = 0\) at \((x_0, y_0, z_0) = (0, 0, 0)\). The formula for the tangent plane in this case is

\[
F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.
\]

We compute the partials:

\[
F_x(x, y, z) = \frac{1}{x + e^z};
\]
\[
F_y(x, y, z) = -2z;
\]
\[
F_z(x, y, z) = \frac{1}{x + e^z} e^z - 2y.
\]

So,

\[
F_x(0, 0, 0) = 1;
\]
\[
F_y(0, 0, 0) = 0;
\]
\[
F_z(0, 0, 0) = 1.
\]

We conclude that the equation of the tangent plane is

\[
x + z = 0.
\]

\[\square\]
5. (10 points) Suppose that $w = x^2 \sin(y) + ye^{xy}$, where $x = s + 2t$ and $y = st$. Use the chain rule to find $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ when $(s, t) = (0, 1)$.

Proof. This is a direct computation by chain rule:

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s}$$

$$= (2x \sin(y) + y^2 e^{xy})(1) + (x^2 \cos(y) + e^{xy} + xye^{xy})(t).$$

and

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t}$$

$$= (2x \sin(y) + y^2 e^{xy})(2) + (x^2 \cos(y) + e^{xy} + xye^{xy})(s).$$

When $(s, t) = (0, 1)$, we have that $(x, y) = (2, 0)$. This means

$$\frac{\partial w}{\partial s} \bigg|_{(s, t)=(0,1)} = 5$$

and

$$\frac{\partial w}{\partial t} \bigg|_{(s, t)=(0,1)} = 0.$$ 

□
6. (10 points) Find the critical points of the function \( f(x, y) = (x^2 + y)e^{y/2} \) and use the second derivative test to classify them. Hint: when finding the critical points, recall that the exponential is always positive. In other words, \( e^c > 0 \) for any value of \( c \in \mathbb{R} \).

Proof. We need to find the (simultaneous) zeros of

\[
f_x(x, y) = 2xe^{y/2}
\]

and

\[
f_y(x, y) = e^{y/2} + (x^2 + y)e^{y/2}\frac{1}{2}.
\]

Solving

\[
f_x(x, y) = 2xe^{y/2} = 0
\]
gives \( x = 0 \). Putting this into the other partial equation

\[
e^{y/2} + (x^2 + y)e^{y/2}\frac{1}{2} = 0
\]
amounts to solving

\[
e^{y/2} + e^{y/2}\frac{y}{2} = 0.
\]

Factoring, we get

\[
e^{y/2}(1 + \frac{y}{2}) = 0,
\]
meaning \( y = -2 \). So, we only have one critical point \((0, -2)\). We compute the second order derivatives to apply the second derivative test:

\[
f_{xx}(x, y) = 2e^{y/2}
\]
\[
f_{xy}(x, y) = xe^{y/2}
\]
\[
f_{yy}(x, y) = e^{y/2}\frac{1}{2} + \frac{1}{2}\left(e^{y/2} + (x^2 + y)e^{y/2}\frac{1}{2}\right).
\]

Plugging in \((x, y) = (0, -2)\), we get

\[
f_{xx}(0, -2) = 2e^{-1}
\]
\[
f_{xy}(0, -2) = 0
\]
\[
f_{yy}(0, -2) = e^{-1}\frac{1}{2}.
\]

So,

\[
f_{xx}(0, -2)f_{yy}(0, -2) - \left(f_{xy}(0, -2)\right)^2 = e^{-2} > 0.
\]

Since \( f_{xx}(0, -2) = 2e^{-1} > 0 \), we conclude that \((0, -2)\) is a local minimum.
7. (10 points) Use Lagrange multipliers to find the maximum and minimum values of 
\( f(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{\pi} \) subject to the constraint \( \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = 1 \).

Proof. We start with the Lagrange equation \( \nabla f = \lambda \nabla g \). After computing the gradients, this amounts to

\[
\begin{pmatrix} -\frac{1}{x^2}, -\frac{1}{y^2}, -\frac{1}{z^2} \end{pmatrix} = \lambda \begin{pmatrix} -\frac{2}{x^3}, -\frac{2}{y^3}, -\frac{2}{z^3} \end{pmatrix}.
\]

In other words,

\[
\begin{align*}
-\frac{1}{x^2} &= -2\lambda \frac{1}{x^3}; \\
-\frac{1}{y^2} &= -2\lambda \frac{1}{y^3}; \\
-\frac{1}{z^2} &= -2\lambda \frac{1}{z^3}.
\end{align*}
\]

Solving, this gives

\[
\begin{align*}
x &= 2\lambda; \\
y &= 2\lambda; \\
z &= 2\lambda.
\end{align*}
\]

So, \( x = y = z \). Putting this back into the constraint, we get

\[
\frac{3}{x^2} = 1,
\]

meaning \( x = \pm \sqrt{3} \). So, the points we need to consider are \( (x, y, z) = (\pm \sqrt{3}, \pm \sqrt{3}, \pm \sqrt{3}) \).

Since

\[
f(\pm \sqrt{3}, \pm \sqrt{3}, \pm \sqrt{3}) = \pm \sqrt{3} + \frac{1}{\pi},
\]

we conclude that

\[
f(\sqrt{3}, \sqrt{3}, \sqrt{3}) = \sqrt{3} + \frac{1}{\pi}
\]

is the maximum value and

\[
f(-\sqrt{3}, -\sqrt{3}, -\sqrt{3}) = -\sqrt{3} + \frac{1}{\pi}
\]

is the minimum value. \(\square\)
8. (10 points) Evaluate $\iint_R x \cos(xy) \, dA$, where $R = [1, 2] \times [0, \pi]$. 

Proof. We have a choice in which variable to integrate first. The natural choice is to integrate the $y$ variable first to avoid integration by parts:

$\iint_R x \cos(xy) \, dA = \int_1^2 \left( \int_0^\pi x \cos(xy) \, dy \right) dx$

$= \int_1^2 \left. \sin(xy) \right|_{y=0}^{y=\pi} \, dx$

$= \int_1^2 \sin(\pi x) \, dx$

$= -\cos(\pi x) \bigg|_{x=2}^{x=1}$

$= 1 - (-\frac{1}{\pi})$

$= -\frac{2}{\pi}$.
9. (15 points) Let $f(x, y) = x^2 + 2y^3$.

(a) (10 points) Find $\iint_D f(x, y) \, dA$, where $D$ is the region in $\mathbb{R}^2$ bounded by the curves $x = 0$, $y = 1$, and $y = x$.

Proof. You could do this either as a type I region or a type II region. As a type I region, this would be

$$\iint_D f(x, y) \, dA = \int_0^1 \int_x^1 x^2 + 2y^3 \, dy \, dx$$

$$= \int_0^1 \left[ \frac{x^2 y}{2} + \frac{2y^4}{4} \right]_{y=x}^{y=1} \, dx$$

$$= \int_0^1 x^2 + 1 - (x^3 + \frac{x^4}{2}) \, dx$$

$$= \left[ \frac{x^3}{3} + \frac{x^2}{2} - \frac{x^4}{4} - \frac{x^5}{10} \right]_{x=0}^{x=1}$$

$$= \frac{1}{3} + \frac{1}{2} - \frac{1}{4} - \frac{1}{10}$$

$$= \frac{29}{60}.$$

As a type II region, this would be

$$\iint_D f(x, y) \, dA = \int_0^1 \int_0^y x^2 + 2y^3 \, dx \, dy$$

$$= \int_0^1 \left[ \frac{x^3}{3} + 2xy^3 \right]_{x=0}^{x=y} \, dy$$

$$= \int_0^1 \frac{y^3}{3} + 2y^4 \, dy$$

$$= \left[ \frac{y^4}{12} + \frac{2y^5}{5} \right]_0^1$$

$$= \frac{1}{12} + \frac{2}{5}$$

$$= \frac{29}{60}.$$

\[\square\]
(b) (5 points) Find the average value of $f(x, y)$ on the region $D$ from part (a).

Proof. One needs to find the area of the region $D$ and divide the answer in part (a) by this area. If you drew a picture, then you would have noticed that the region $D$ is just a right triangle with two sides of length 1, and so the area is $\frac{1}{2}$. You could also compute the iterated integral as a type I region:

$$\int\int_D 1 \, dA = \int_0^1 \int_x^1 1 \, dy \, dx$$

$$= \int_0^1 (1 - x) \, dy$$

$$= \left( x - \frac{x^2}{2} \right) \bigg|_{x=0}^{x=1}$$

$$= \frac{1}{2}.$$

Alternatively, as a type II region, this amounts to

$$\int\int_D 1 \, dA = \int_0^1 \int_0^y 1 \, dx \, dy$$

$$= \int_0^1 y \, dy$$

$$= \frac{y^2}{2} \bigg|_{y=0}^{y=1}$$

$$= \frac{1}{2}.$$

In any case, the average value is

$$\frac{\int\int_D f(x, y) \, dA}{\int\int_D 1 \, dA} = \frac{29}{60} \cdot \frac{1}{2} = \frac{29}{30}.$$

□
(ADDITIONAL SPACE FOR WORK, CLEARLY INDICATE THE PROBLEM YOU ARE WORKING ON)
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