MATH 10C - MIDTERM #1

Name (Last, First): ____________________________________________

Student ID: __________________________

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT.

THIS EXAM WILL BE SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS ON THE PAPER PROVIDED. DO NOT REMOVE ANY OF THE PAGES.

THE EXAM CONSISTS OF 4 QUESTIONS. YOUR ANSWERS SHOULD BE CAREFULLY JUSTIFIED.
1. (20 points)
   (a) (15 points) Find a unit vector orthogonal to the plane that contains the points
   \( P = (1, 2, 3) \), \( Q = (4, -1, 2) \), and \( R = (-2, 3, 1) \).

   Find an equation for this plane.

   \[
   \overrightarrow{PQ} = \langle 4 - 1, -1 - 2, 2 - 3 \rangle = \langle 3, -3, -1 \rangle
   \]

   \[
   \overrightarrow{PR} = \langle -2 - 1, 3 - 2, 1 - 3 \rangle = \langle -3, 1, -2 \rangle
   \]

   \[
   \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix}
   \hat{i} & \hat{j} & \hat{k} \\
   3 & -3 & -1 \\
   -3 & 1 & -2
   \end{vmatrix} = \hat{i}((-3)(-2) - (-1)(1)) - \hat{j}((-3)(1) - (-1)(-3)) + \hat{k}((3)(1) - (1)(-3))
   \]

   \[
   = 7\hat{i} + 9\hat{j} - 6\hat{k}
   \]

   a unit vector is \( \frac{7\hat{i} + 9\hat{j} - 6\hat{k}}{\sqrt{7^2 + 9^2 + (-6)^2}} \)

   an equation for the plane is

   \[
   7(x - 1) + 9(y - 2) - 6(z - 3) = 0
   \]
(b) (5 points) Find the area of the triangle with vertices $P$, $Q$, and $R$.

\[ \text{the area is } \frac{\| \overrightarrow{PQ} \times \overrightarrow{PR} \|}{2} \]

\[ = \frac{\sqrt{7^2 + 9^2 + (-6)^2}}{2} \]
2. (10 points) Determine if the vectors 
\[ u = (0, 1, 2), \quad v = (1, 2, 3), \quad \text{and} \quad w = (2, 3, 4) \]
are co-planar.

\[
\begin{vmatrix}
0 & 1 & 2 \\
1 & 2 & 3 \\
2 & 3 & 4
\end{vmatrix} = 0 ((2)(4) - (3)(3)) \\
-1 ((1)(4) - (3)(2)) \\
+ 2 ((1)(3) - (2)(2))
\]

\[ = 0 + 2 - 2 = 0 \]

Yes, co-planar
3. (10 points) Determine the angle \( \theta \in \left[ 0, \frac{\pi}{2} \right] \) of intersection between the planes
\[
2x + 3y + z = \sqrt{\pi}
\]
and
\[
x - 2y + z - \pi^2 = 0.
\]

\[
\vec{n}_1 = \langle 2, 3, 1 \rangle
\]
\[
\vec{n}_2 = \langle 1, -2, 1 \rangle
\]

\[
\vec{n}_1 \cdot \vec{n}_2 = (2)(1) + (3)(-2) + (1)(1) = 2 - 6 + 1 = -3
\]

\[
\cos(\Theta) = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1||\vec{n}_2|} = \frac{1 - 3}{\sqrt{2^2 + 3^2 + 1^2} \sqrt{1^2 + (-2)^2 + 1^2}}
\]

\[
\Theta = \cos^{-1}\left(\frac{3}{\sqrt{2^2 + 3^2 + 1^2} \sqrt{1^2 + (-2)^2 + 1^2}}\right)
\]
4. (10 points) Determine whether or not the planes

\[ x - 2(y - 1) = 3z \]

and

\[ -2(x - 1) = 1 - 4y - 6z \]

are parallel. If parallel, find the distance between them. If not parallel, find the line of intersection between them.

\[ x - 2(y - 1) = 3z \quad \text{is the same as} \]

\[ x - 2(y - 1) - 3z = 0 \quad \text{so} \quad \mathbf{n}_1 = <1, -2, -3> \]

\[ -2(x - 1) = 1 - 4y - 6z \quad \text{is the same as} \]

\[ -2(x - 1) + 4y + 6z - 1 = 0 \quad \text{so} \quad \mathbf{n}_2 = <-2, 4, 6> \]

Since \( \mathbf{n}_2 = -2\mathbf{n}_1 \), the two planes are parallel.

\((0, 1, 0)\) is clearly a point on the plane \( x - 2(y - 1) = 3z \).

The distance to the second plane is then

\[
\frac{|(-2)(0) + 4(1) + 6(0) - 1|}{\sqrt{(-2)^2 + 4^2 + 6^2}} = \frac{5}{\sqrt{(-2)^2 + 4^2 + 6^2}}
\]
(ADDITIONAL SPACE FOR WORK, CLEARLY INDICATE THE PROBLEM YOU ARE WORKING ON)
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