MATH 10C - MIDTERM #2

Name (Last, First):

Student ID:

**REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT.**

**THIS EXAM WILL BE SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS ON THE PAPER PROVIDED. DO NOT REMOVE ANY OF THE PAGES.**

**THE EXAM CONSISTS OF 4 QUESTIONS. YOUR ANSWERS SHOULD BE CAREFULLY JUSTIFIED.**
1. (20 points) Let $f(x, y, z) = xe^y + ye^z + ze^x$.

(a) (10 points) Find the gradient of the function $f$ at the point $(0, 0, 0)$. Find the directional derivative of the function $f$ at the point $(0, 0, 0)$ in the direction $\vec{v} = (1, 2, 3)$.

\[
\nabla f(x, y, z) = \left< x, y, z \right> = \left< e^y + ze^x, xe^y + e^z, ye^z + e^x \right>
\]

\[
\nabla f(0, 0, 0) = \left< 1, 1, 1 \right>
\]

Let $\vec{u} = \frac{\vec{v}}{||\vec{v}||}$

Then we want

\[
D_{\vec{u}} f(0, 0, 0) = \nabla f(0, 0, 0) \cdot \frac{\vec{v}}{||\vec{v}||}
\]

\[
= \left< 1, 1, 1 \right> \cdot \left< \frac{1}{\sqrt{1^2 + 2^2 + 3^2}}, \frac{2}{\sqrt{1^2 + 2^2 + 3^2}}, \frac{5}{\sqrt{1^2 + 2^2 + 3^2}} \right>
\]

\[
= \frac{6}{\sqrt{1^2 + 2^2 + 3^2}}
\]
(b) (10 points) Find the direction of the maximal rate of decrease for the function $f$ at the point $(0, 0, 0)$. What is this value of this decrease?

$$D_{\mathbf{u}} f(0, 0, 0) = \langle 1, 1, 1 \rangle \cdot \mathbf{u} = \| \langle 1, 1, 1 \rangle \| \cdot \| \mathbf{u} \| \cdot \cos(\theta)$$

$$= \| \langle 1, 1, 1 \rangle \| \cdot \cos(\theta)$$

Want this to be as small as possible,

so $\theta = \pi$

The only vector of unit length that makes an angle of $\pi$ with $\langle 1, 1, 1 \rangle$ is

$$\frac{-\langle 1, 1, 1 \rangle}{\| \langle 1, 1, 1 \rangle \|} = \left\langle \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} \right\rangle = \mathbf{u}$$

The value of this decrease is

$$\langle 1, 1, 1 \rangle \cdot \mathbf{u} = \langle 1, 1, 1 \rangle \cdot \left( \frac{-\langle 1, 1, 1 \rangle}{\| \langle 1, 1, 1 \rangle \|} \right)$$

$$= -\frac{\| \langle 1, 1, 1 \rangle \|^2}{\| \langle 1, 1, 1 \rangle \|} = -\| \langle 1, 1, 1 \rangle \|$$

$$= -\sqrt{3}$$
2. (10 points) Use the Chain Rule to find the partial derivatives $\frac{\partial R}{\partial u}$ and $\frac{\partial R}{\partial v}$ for 

$$R = xe^{yz^3},$$

where $x = 2uv$, $y = u - v$, and $z = u + v$.

$$\frac{\partial x}{\partial u} = 2v \quad \frac{\partial x}{\partial u} = 1 \quad \frac{\partial x}{\partial u} = 1$$
$$\frac{\partial x}{\partial v} = 2u \quad \frac{\partial x}{\partial u} = -1 \quad \frac{\partial x}{\partial v} = 1$$

$$\frac{\partial R}{\partial x} = e^{yz^3} \quad \frac{\partial R}{\partial y} = xe^{yz^3} \quad \frac{\partial R}{\partial z} = xe^{yz^3}(3yz^2)$$

$$\frac{\partial R}{\partial u} = \frac{\partial R}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial R}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial R}{\partial z} \frac{\partial z}{\partial u}$$
$$= (e^{yz^3})(2v) + (xe^{yz^3}(2yz^3))(1) + (xe^{yz^3}(3yz^2))(1)$$

$$\frac{\partial R}{\partial v} = \frac{\partial R}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial R}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial R}{\partial z} \frac{\partial z}{\partial v}$$
$$= (e^{yz^3})(2u) + (xe^{yz^3}(2yz^3))(-1) + (xe^{yz^3}(3yz^2))(1)$$
3. (10 points) Find the tangent plane to the function \( f(x, y) = \sqrt{x + e^{4y}} \) at the point (3, 0).

\[
\begin{align*}
  f'(3, 0) &= \sqrt{3 + 1} = 2 \\
  f_x(x, y) &= \frac{1}{2} (x + e^{4y})^{-1/2} \\
  f_y(x, y) &= \frac{1}{2} (x + e^{4y})^{-1/2} (e^{4y})(4) \\
  f_x(3, 0) &= \frac{1}{2} (3 + 1)^{-1/2} = \frac{1}{4} \\
  f_y(3, 0) &= \frac{1}{2} (3 + 1)^{-1/2} (1)(4) = 1 \\
\end{align*}
\]

Plane: \[ z - 2 = \frac{1}{4} (x - 3) + 1(y - 0) \]
4. (10 points) Suppose that \( \mathbf{r}(t) = \langle t, e^t, te^t \rangle \). What is the domain of this function? Find the unit tangent vector to this curve at time \( t = 0 \).

The domain of all three coordinate functions \( t, e^t, te^t \) is \( \mathbb{R} \), so the domain of \( \mathbf{r}(t) \) is also \( \mathbb{R} \).

\[
\mathbf{r}'(t) = \langle 1, e^t, te^t + e^t \rangle
\]

\[
\mathbf{r}'(0) = \langle 1, 1, 1 \rangle
\]

The unit vector is

\[
\text{unit vector is } \frac{\mathbf{r}'(0)}{|| \mathbf{r}'(0) ||} = \frac{\langle 1, 1, 1 \rangle}{\sqrt{3}}
\]

\[
= \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle
\]
(ADDITIONAL SPACE FOR WORK, CLEARLY INDICATE THE PROBLEM YOU ARE WORKING ON)
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