MATH 10C - MIDTERM #1

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1. (15 points) Find *two* unit vectors orthogonal to the plane that contains the points

$$P = (0,1,2), \quad Q = (2,-1,2), \quad \text{and} \quad R = (-1,0,1).$$

Also, find an equation for this plane.

Solution. Note that $\overrightarrow{PQ} = \langle 2,-2,0\rangle$ and $\overrightarrow{PR} = \langle -1,-1,-1\rangle$. We then compute the cross product

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 0 \\ -1 & -1 & -1 \end{vmatrix} = \begin{vmatrix} -2 & 0 \\ -1 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 0 \\ -1 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -2 \\ -1 & -1 \end{vmatrix} \mathbf{k} = 2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}.$$

This gives a vector orthogonal to the plane that contains the points P, Q, and R; however, this is not a unit vector. So, we divide by the length to get

$$\mathbf{u} = \frac{2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}}{\sqrt{(2)^2 + (2)^2 + (-4)^2}} = \frac{2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}}{\sqrt{24}} = \left\langle \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}} \right\rangle.$$

A second unit vector is obtained by (scalar) multiplying by -1:

$$\mathbf{v} = -\mathbf{u} = \left\langle \frac{-1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right\rangle.$$

We use the original orthogonal vector and the point P to write the equation of the plane:

$$2(x-0) + 2(y-1) - 4(z-2) = 0.$$

□

2. (10 points) Is the triangle with vertices $P = (0, 1, 2)$, $Q = (2, -1, 2)$, and $R = (-1, 0, 1)$ right-angled? (Note: the points P, Q, and R are the same points from **1.**)

Solution. Note that $\overrightarrow{PQ} \cdot \overrightarrow{PR} = \langle 2, -2, 0\rangle \cdot \langle -1, -1, -1\rangle = -2 + 2 + 0 = 0$. This means that two of the sides of our triangle are perpendicular, and so our triangle is right-angled. □

3. (10 points) Determine if the vectors

$$\mathbf{u} = \langle 1, -1, 2 \rangle, \quad \mathbf{v} = \langle 2, 0, 3 \rangle, \quad \text{and} \quad \mathbf{w} = \langle 3, -1, 5 \rangle$$

are co-planar.

Solution. We compute the scalar triple product

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 0 & 3 \\ 3 & -1 & 5 \end{vmatrix} = \begin{vmatrix} 0 & 3 \\ -1 & 5 \end{vmatrix} 1 - \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} (-1) + \begin{vmatrix} 2 & 0 \\ 3 & -1 \end{vmatrix} 2 = 3 + 1 - 4 = 0.$$

This means that the volume of the parallelepiped determined by the vectors **u**, **v**, and **w** has volume 0, which is only possible if it is degenerate (flat) and is contained in a plane. So, the vectors **u**, **v**, and **w** are co-planar. □

4. (15 points) Consider the planes

$$-x - 3(y-2) + 2(z+3) = 0$$

and

$$2x + 6(y-1) - 4(z-4) = 0.$$

Determine whether or not the planes are parallel. If they are parallel, find the distance between them. If they are not parallel, find the line of intersection between them.

Solution. Our planes have normal vectors $\mathbf{n}_1 = \langle -1, -3, 2\rangle$ and $\mathbf{n}_2 = \langle 2, 6, -4\rangle$ respectively, which are multiples of each other (for example, $-2\mathbf{n}_1 = \mathbf{n}_2$). So, they are parallel. To compute the distance between them, we pick a point on the first plane. An obvious point (based on the form of the equation) is $(0, 2, -3)$. We then rewrite the second plane in the form that allows us to apply the distance equation:

$$2x + 6(y-1) - 4(z-4) = 0 \iff 2x + 6y - 4z + 10 = 0$$

So, the distance between the planes is

$$\frac{|2(0) + 6(2) - 4(-3) + 10|}{\sqrt{(2)^2 + (6)^2 + (-4)^2}} = \frac{17}{\sqrt{14}}$$

□