MATH 10C - MIDTERM #2

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT.

THE EXAM CONSISTS OF 4 QUESTIONS. YOUR ANSWERS MUST BE CAREFULLY JUSTIFIED TO RECEIVE CREDIT.

IF YOU DO NOT ASSIGN THE PAGES OF YOUR WORK TO THE QUESTIONS OF THE EXAM IN YOUR UPLOAD TO GRADESCOPE, YOU WILL LOSE POINTS (1 POINT FOR EVERY QUESTION THAT YOU FAIL TO ASSIGN THE PAGES TO).

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You DO NOT need to copy the questions down. Your upload only needs to contain your work (with the pages properly assigned in Gradescope).

1. (20 points) Let $f(x,y,z) = (x+y+z)e^{xy^2z^3}$.

(a) (10 points) Find the gradient of the function f at the point $(0,0,0)$. Find the directional derivative of the function f at the point $(0,0,0)$ in the direction $\vec{v} = \langle 1,2,3 \rangle$.

Solution.

$\nabla f(x,y,z)$

$= \langle e^{xy^2z^3} + (x+y+z)e^{xy^2z^3}(y^2z^3), e^{xy^2z^3} + (x+y+z)e^{xy^2z^3}(2xyz^3), e^{xy^2z^3} + (x+y+z)e^{xy^2z^3}(3xy^2z^2) \rangle$

so

$$\nabla f(0,0,0) = \langle 1,1,1 \rangle.$$

The unit vector corresponding to the direction of $\vec{v}$ is

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \left\langle \frac{1}{\sqrt{1^2+2^2+3^2}}, \frac{2}{\sqrt{1^2+2^2+3^2}}, \frac{3}{\sqrt{1^2+2^2+3^2}} \right\rangle.$$

So,

$$D_{\vec{u}} f(0,0,0) = \vec{u} \cdot \nabla f(0,0,0) = \frac{6}{\sqrt{1^2+2^2+3^2}}.$$

□

(b) (10 points) Find the unit vector in the direction of the maximal rate of *decrease* for the function f at the point $(0,0,0)$. What is the value of the directional derivative in this direction?

Solution. We know that $-\nabla f(0,0,0)$ points in the direction of the maximal rate of decrease. The corresponding unit vector is

$$\frac{-\nabla f(0,0,0)}{|-\nabla f(0,0,0)|} = \left\langle \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} \right\rangle.$$

We also know that the directional directive in this direction is simply

$$D_{\frac{-\nabla f(0,0,0)}{|-\nabla f(0,0,0)|}} f(0,0,0) = -|\nabla f(0,0,0)| = -\sqrt{3}.$$

□

2. (10 points) Use the chain rule to find the partial derivative

$$\frac{\partial R}{\partial v}$$

for

$$R = (x + yz^2)^3,$$

where

$$x = u + v, \quad y = u - v, \quad \text{and} \quad z = 2uv.$$

You may leave your answer as a product of terms, but your answer should not have any derivative operations remaining to be performed. Your final answer should only be a function of u and v.

Solution. By the chain rule,

$$\frac{\partial R}{\partial v} = \frac{\partial R}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial R}{\partial y}\frac{\partial y}{\partial v} + + \frac{\partial R}{\partial z}\frac{\partial z}{\partial v}.$$

So, we compute

$$\frac{\partial R}{\partial x} = 3(x + yz^2)^2 = 3((u + v) + (u - v)(2uv))^2$$
$$\frac{\partial R}{\partial y} = 3(x + yz^2)^2(z^2) = 3((u + v) + (u - v)(2uv))^2(2uv)^2$$
$$\frac{\partial R}{\partial z} = 3(x + yz^2)^2(2yz) = 3((u + v) + (u - v)(2uv))^2(2(u - v)(2uv))$$
$$\frac{\partial x}{\partial v} = 1$$
$$\frac{\partial y}{\partial v} = -1$$
$$\frac{\partial z}{\partial v} = 2u.$$

So, our final answer is

$$3((u+v)+(u-v)(2uv))^2 - 3((u+v)+(u-v)(2uv))^2(2uv)^2 + (2u)(3((u+v)+(u-v)(2uv))^2(2(u-v)(2uv))).$$

□

3. (10 points) Find the tangent plane to the function $f(x, y) = \sqrt{xy^2 + \ln(x) + 1}$ at the point $(1, 0)$. Hint: recall that $\ln(1) = 0$.

Solution. Note that $x_0 = 1$, $y_0 = 0$, and $z_0 = f(x_0, y_0) = \sqrt{0 + 0 + 1} = 1$. Moreover,

$$f_x(x, y) = \frac{1}{2}(xy^2 + \ln(x) + 1)^{-1/2}(y^2 + \frac{1}{x}), \quad f_y(x, y) = \frac{1}{2}(xy^2 + \ln(x) + 1)^{-1/2}(2xy).$$

So, $f_x(1, 0) = \frac{1}{2}$ and $f_y(1, 0) = 0$. Using our formula $f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = z - z_0$ for the tangent plane, we get

$$\frac{1}{2}(x - 1) = z - 1.$$

□

4. (10 points) Suppose that $\mathbf{r}(t) = \langle e^{\cos(t)}, \sin(1 - e^{-t}), t\rangle$. Find the unit tangent vector to this curve at time $t = 0$.

Solution. Note that

$$\mathbf{r}'(t) = \langle e^{\cos(t)}(-\sin(t)), \cos(1 - e^{-t})(-e^{-t})(-1), 1\rangle.$$

So,

$$\mathbf{r}'(0) = \langle 0, 1, 1\rangle.$$

The unit tangent vector is then

$$\frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \left\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle.$$

□