The chain rule

Single variable case:

$y = f(x)$ and $x = g(t)$

f: differentiable $\quad$ g: differentiable

$$\begin{array}{c} y \\ | \\ x \\ | \\ t \end{array}$$

so $y = f(g(t))$ and

$$\frac{dy}{dt} = f'(g(t))\,g'(t) = f'(x)\,g'(t) = \frac{dy}{dx}\frac{dx}{dt}$$

Multivariable (case 1):

$z = f(x, y)$

differentiable

$x = g(t)$, $\quad y = h(t)$

differentiable $\quad$ differentiable

$$\begin{tikzcd} & z \arrow[dl, no head] \arrow[dr, no head] & \\ x \arrow[d, no head] & & y \arrow[d, no head] \\ t & & t \end{tikzcd}$$

then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

or $\frac{\partial f}{\partial x}$ $\quad$ or $\frac{\partial f}{\partial y}$

Example: $z = ye^{x} + \sin(xy)$

$x = \cos(t)$, $y = \sin(t)$

find $\frac{dz}{dt}$ when $t = 0$

Ans:

Example: $z = ye^{x} + \sin(xy)$

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Ans: $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

$$= (ye^{x} + y\cos(xy))(-\sin(t)) + (e^{x} + x\sin(y))(\cos(t))$$

when $t=0$, $x=1$ and $y=0$ so

$$\frac{dz}{dt} = 0 + e = e$$

Example: the pressure P (in kilopascals), volume V (in liters), and temperature T (in kelvins) of a mole of an ideal gas are related by $PV = 8.31T$

find the rate at which the pressure is changing when the temperature is 250 K and increasing at a rate .1 K/s and the volume is 100 L and increasing at a rate .5 L/s

Example: $PV = 8.31T$

Find the rate at which the pressure is changing when the temperature is 250 K and increasing at a rate .1 K/s and the volume is 100 L and increasing at a rate .5 L/s

Ans: $P = \frac{8.31T}{V} = f(T, V)$

$$\frac{dP}{dt} = \frac{\partial P}{\partial T}\frac{dT}{dt} + \frac{\partial P}{\partial V}\frac{dV}{dt} = \frac{8.31}{V}\frac{dT}{dt} - \frac{8.31T}{V^2}\frac{dV}{dt}$$

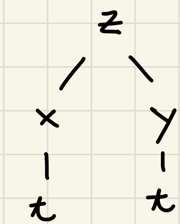
$$= \frac{8.31}{100}(.1) - \frac{8.31(250)}{(100)^2}(.5) = -.095565 \text{ kPa/s}$$

Multivariable chain rule

Multivariable (case 1):

$z = f(x, y)$ — differentiable

$x = g(t)$ — differentiable, $y = h(t)$ — differentiable



then $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

or $\frac{\partial f}{\partial x}$ or $\frac{\partial f}{\partial y}$

(case 2):

$$z = f(x, y)$$

$$x = g(s, t), \quad y = h(s, t)$$

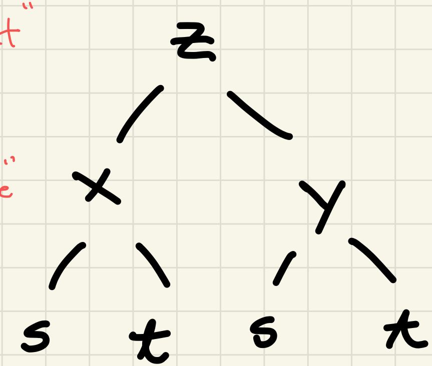
then $$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

"dependent"

"intermediate"

"independent"



Example: $z = e^{x} \ln(y)$

$x = s^2 + t^3 \qquad y = s^2 t^3$

find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$

Ans:

Example: $z = e^x \ln(y)$

$x = s^2 + t^3 \qquad y = s^2 t^3$

find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$

Ans:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \left(e^x \ln(y)\right)(2s) + \left(\frac{e^x}{y}\right)(2st^3)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \left(e^x \ln(y)\right)(3t^2) + \left(\frac{e^x}{y}\right)(3s^2t^2)$$

Multivariable chain rule (general case):

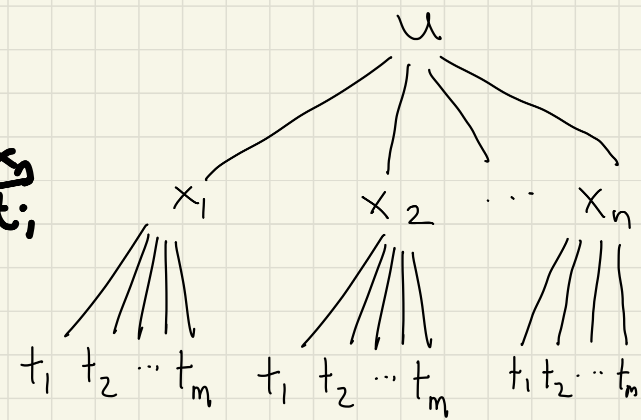
$$u = f(x_1, x_2, \dots, x_n)$$

$$x_j = g_j(t_1, t_2, \dots, t_m)$$

j → some number in $\{1, \dots, n\}$

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

i → some number in $\{1, \dots, m\}$

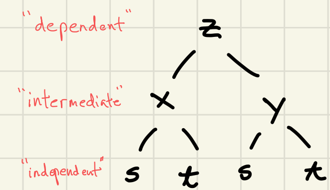


Multivariable chain rule (case 2):

$z = f(x, y)$

$x = g(s,t)$, $y = h(s,t)$

then
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$



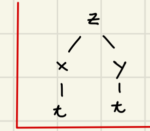
Multivariable (case 1):

$z = f(x,y)$ ← differentiable

$x = g(t)$ ← differentiable, $y = h(t)$ ← differentiable

then $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$

$\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$, $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$



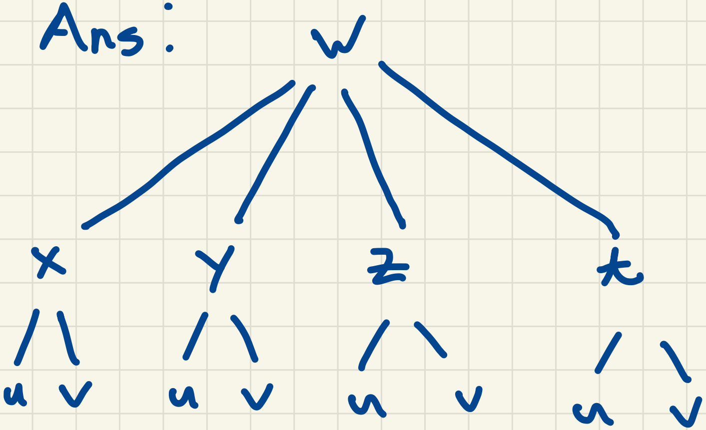
Example: write out the chain rule for the case $w = f(x, y, z, t)$ and $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$, and $t = t(u,v)$

(i.e, find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$)

Ans:

Example: write out the chain rule for the case $w = f(x, y, z, t)$ and $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$, and $t = t(u,v)$ (i.e, find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$)

Ans:



$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial v}$$

Example: $u = x^3y^2 + \ln(z)e^x$

$x = \frac{r}{s}\cos(t)$, $y = \sqrt{rs+t}$, $z = r^2 s e^t$

find $\frac{\partial u}{\partial r}$ when $r=2$, $s=1$, $t=0$

Ans:

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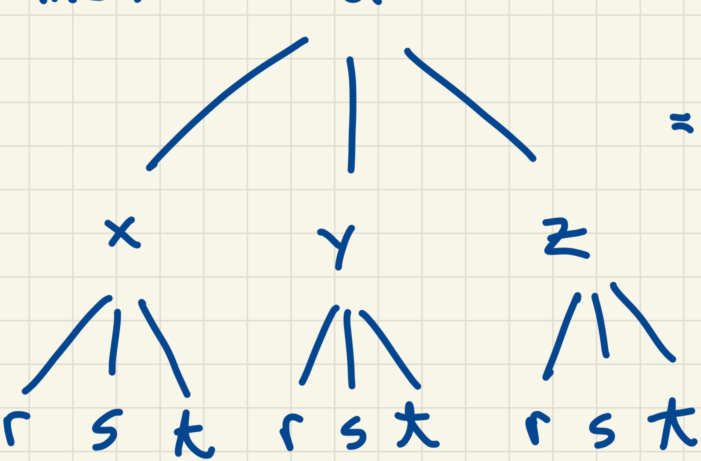

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r}$$

$$= \left(3x^2y^2 + \ln(z)e^x\right)\frac{\cos(t)}{s} + (2x^3y)\left(\frac{s}{2\sqrt{rs+t}}\right)$$

$$+ \left(\frac{e^x}{z}\right)\left(2rse^t\right)$$

Example: $u = x^3y^2 + \ln(z)e^x$

$$x = \frac{r}{s}\cos(t), \quad y = \sqrt{rs+t}, \quad z = r^2 s e^t$$

find $\frac{\partial u}{\partial r}$ when $r = 2, \ s = 1, \ t = 0$

$x = 2$
$y = \sqrt{2}$
$z = 4$

Ans: $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r}$

$$= \left(3x^2y^2 + \ln(z)e^x\right)\frac{\cos(t)}{s} + (2x^3y)\left(\frac{s}{2\sqrt{rs+t}}\right) + \left(\frac{e^x}{z}\right)\left(2rse^t\right)$$

so $\frac{\partial u}{\partial r} = 24 + \ln(4)e^2 + 8 + e^2 = 32 + e^2(\ln(4) + 1)$

Example: $g(s,t) = f(s^2 - t^2, t^2 - s^2)$

Show that g satisfies the equation

$$t \frac{\partial g}{\partial s} + s \frac{\partial g}{\partial t} = 0$$

Ans:

Example: $g(s,t) = f(s^2-t^2,\ t^2-s^2)$

show that g satisfies the equation

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$$

Ans: let $x = x(s,t) = s^2 - t^2$

$y = y(s,t) = t^2 - s^2$

then
$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x}(2s) + \frac{\partial f}{\partial y}(-2s)$$
$$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial f}{\partial x}(-2t) + \frac{\partial f}{\partial y}(2t)$$

Example: $z = f(x, y)$, $x = r^2 + s^3$, $y = 2rs$

find $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial^2 z}{\partial r^2}$

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Functions of x and y

Example: $z = f(x,y)$, $x = r^2 + s^3$, $y = 2rs$

find $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial^2 z}{\partial r^2}$

Functions of x and y

Ans: $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial r} = \dfrac{\partial z}{\partial x}(2r) + \dfrac{\partial z}{\partial y}(2s)$

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left[\frac{\partial z}{\partial r}\right] = \frac{\partial z}{\partial x}(2) + \left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y \partial x}\frac{\partial y}{\partial r}\right)(2r)$$

$$+ \frac{\partial z}{\partial y}(0) + \left(\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial r}\right)(2s)$$

$$= 2\frac{\partial z}{\partial x} + 4r^2\frac{\partial^2 z}{\partial x^2} + 4rs\frac{\partial^2 z}{\partial y \partial x} + 4rs\frac{\partial^2 z}{\partial x \partial y} + 4s^2\frac{\partial^2 z}{\partial y^2}$$

Implicit differentiation

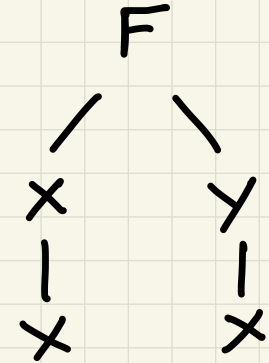
Suppose we have an equation of the form $F(x,y)=0$ that defines y implicitly as a function of x

(simple example: $F(x,y) = y - x$)

Call this function $y = f(x)$

then $F(x,y) = F(x, f(x)) = 0$

$$F(x,y) = F(x, f(x)) = 0$$



Chain rule says

$$\frac{\partial F}{\partial x}\underbrace{\frac{dx}{dx}}_{1} + \frac{\partial F}{\partial y}\frac{\partial y}{dx} = 0$$

so $$\frac{dy}{dx} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = \frac{-F_x}{F_y}$$

Note: need some techical conditions, but assume okay in this class

Example: find y' if $2x^2 + y^4 = 6xy$

Ans:

Example: find y' if $2x^2 + y^4 = 6xy$

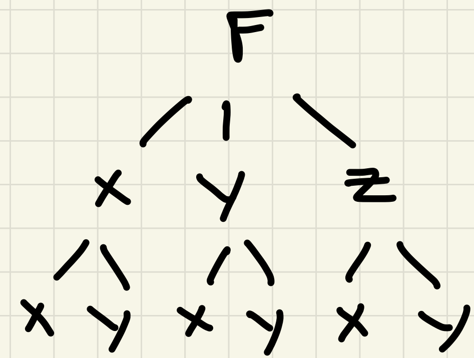
Ans: $F(x,y) = 2x^2 + y^4 - 6xy = 0$

$$\frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-(4x - 6y)}{4y^3 - 6x}$$

More variables: assume $F(x,y,z)=0$

defines z as a function of x and y

call this function $z = f(x,y)$



then $F(x,y,f(x,y))=0$

$$\frac{\partial F}{\partial x}\overset{1}{\frac{dx}{dx}} + \frac{\partial F}{\partial y}\overset{0}{\frac{dy}{dx}} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0 \quad \rightarrow \quad \frac{\partial z}{\partial x} = \frac{-F_x}{F_z}$$

$$\frac{\partial F}{\partial y}\overset{0}{\frac{dx}{dy}} + \frac{\partial F}{\partial y}\overset{1}{\frac{dy}{dy}} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0 \quad \rightarrow \quad \frac{\partial z}{\partial y} = \frac{-F_y}{F_z}$$

Example: find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

$$x^2 + y^3 + z^5 + 7xyz = \pi$$

Ans:

Example: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

$$x^2 + y^3 + z^5 + 7xyz = \pi$$

Ans: $F(x, y, z) = x^2 + y^3 + z^5 + 7xyz - \pi = 0$

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = -\frac{2x + 7yz}{5z^4 + 7xy}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = -\frac{3y^2 + 7xz}{5z^4 + 7xy}$$