Cross product

Last time: dot product $\vec{a} \cdot \vec{b} \in \mathbb{R}$

(informally, measures the extent to which $\vec{a}$ and $\vec{b}$ point in the same direction)

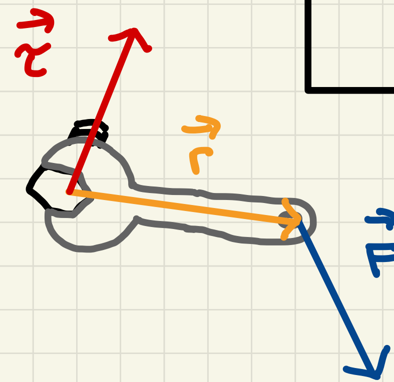
Today: cross product $\vec{a} \times \vec{b}$

- is a vector perpendicular to $\vec{a}$ and $\vec{b}$
- natural geometric interpretation

Torque

Note: $\vec{\tau} \perp \vec{r}$

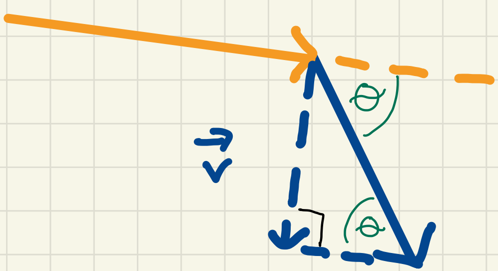
$\vec{\tau} \perp \vec{F}$



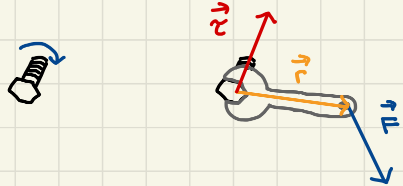
$|\vec{\tau}|$ depends on $|\vec{r}|$

$|\vec{F}|$

AND angle θ between $\vec{r}$ and $\vec{F}$

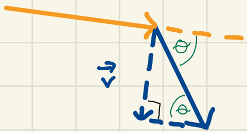


the length $|\vec{v}| = |\vec{F}|\sin(\theta)$



$|\vec{\tau}|$ depends on $|\vec{r}|$

$|\vec{F}|$



AND angle θ between $\vec{r}$ and $\vec{F}$

the length $|\vec{v}| = |\vec{F}|\sin(\theta)$

So, the magnitude of our torque is

$$|\vec{\tau}| = |\vec{r}||\vec{F}|\sin(\theta)$$

What about the direction?

Let $\vec{n}$ be a unit vector in the direction in which the right-handed bolt moves

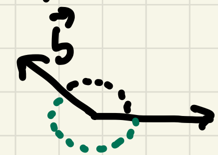
Then $$\vec{r} \times \vec{F} = \underbrace{\left(|\vec{r}||\vec{F}|\sin(\theta)\right)}_{\text{magnitude}} \underbrace{\vec{n}}_{\text{direction}}$$

This is the cross product

General definition of the cross product

If $\vec{a}, \vec{b}$ are vectors in $\mathbb{R}^3$,

then $\vec{a} \times \vec{b} = \left(|\vec{a}|\,|\vec{b}|\sin(\theta)\right)\vec{n}$,

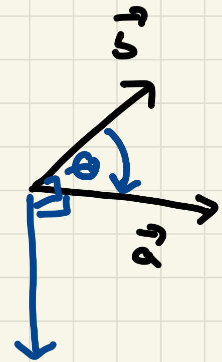
where θ is the angle between $\vec{a}$ and $\vec{b}$
with $\theta \in [0, \pi]$ (recall 

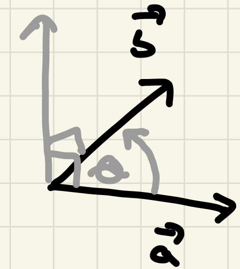
) $\vec{b} \times \vec{a}$

and $\vec{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ and whose direction is given by the right-hand rule: curl your fingers through the angle from $\vec{a}$ to $\vec{b}$ (NOT $\vec{b}$ to $\vec{a}$), then your thumb points in the direction of $\vec{n}$

$$\boxed{\vec{a}\times\vec{b} = \left(|\vec{a}|\,|\vec{b}|\sin(\theta)\right)\vec{n}}$$

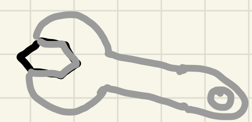
Immediate properties : $\vec{b}\times\vec{a} = -\,\vec{a}\times\vec{b}$





Again, $\vec{a}\times\vec{b} \perp \vec{a}$

$\vec{a}\times\vec{b} \perp \vec{b}$

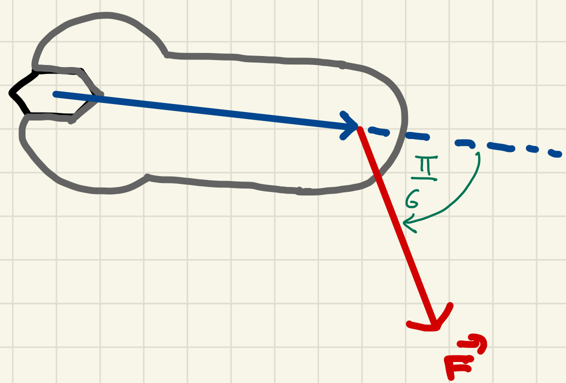


If $\vec{a}, \vec{b} \neq \vec{0}$, then

$\vec{a}$ and $\vec{b}$ are parallel $\iff$ $\vec{a}\times\vec{b} = \vec{0}$

Example

A bolt is tightened by applying a 100 N force to a .1 m wrench as below



Find the magnitude of the torque

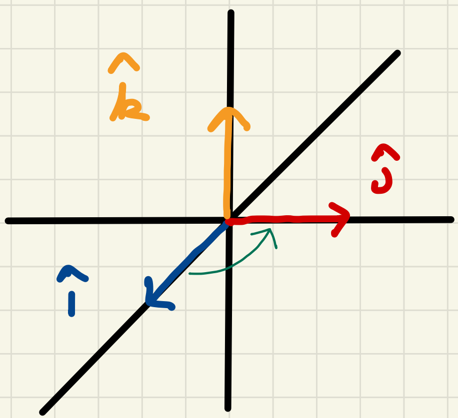
$$|\tau| = (100\text{ N})(.1\text{ m})\sin\left(\frac{\pi}{6}\right) = 5\text{ N}\cdot\text{m}$$

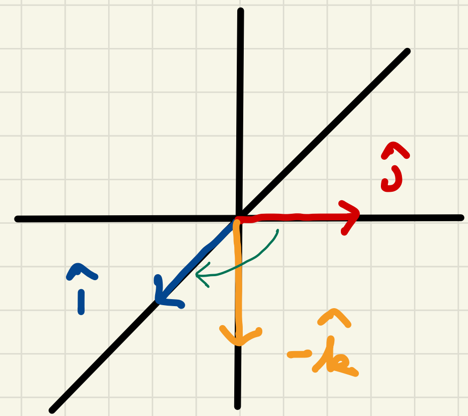
Standard basis vectors

$\langle a_1, a_2, a_3 \rangle$
$= a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$\hat{\imath} \times \hat{\jmath} = \hat{k}$

$\hat{\jmath} \times \hat{\imath} = -\hat{k}$





Exercise:

$\hat{\imath} \times \hat{k} = -\hat{\jmath}$

$\hat{k} \times \hat{\imath} = \hat{\jmath}$

$\hat{\jmath} \times \hat{k} = \hat{\imath}$

$\hat{k} \times \hat{\jmath} = -\hat{\imath}$

Some warnings

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

$$(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c}) \quad \text{in general}$$

example: $(\hat{\imath} \times \hat{\imath}) \times \hat{\jmath} = \vec{0} \times \hat{\jmath} = \vec{0}$

$$\hat{\imath} \times (\hat{\imath} \times \hat{\jmath}) = \hat{\imath} \times \hat{k} = -\hat{\jmath} \neq \vec{0}$$

in other words, not associative

Properties of the cross product

You will not be responsible for the proofs, but we need these properties

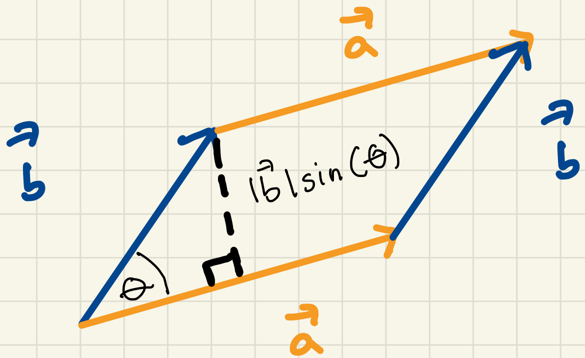
1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

2. $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$

3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

4. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

Geometric interpretation



Area of the parallelogram is $|\vec{a}||\vec{b}|\sin(\theta)$, which is the magnitude of the cross product $\vec{a}\times\vec{b}$

In other words, $|\vec{a}\times\vec{b}|$ is the area of the parallelogram (note $|\vec{a}\times\vec{b}| = |\vec{b}\times\vec{a}|$)

$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$

Intermission

Components

$$\langle a_1, a_2, a_3\rangle \cdot \langle b_1, b_2, b_3\rangle = a_1b_1 + a_2b_2 + a_3b_3$$

$$\langle a_1, a_2, a_3\rangle \times \langle b_1, b_2, b_3\rangle = ?$$

Rewrite $\langle a_1, a_2, a_3\rangle = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$\langle b_1, b_2, b_3\rangle = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

$$(a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}) \times (b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}) = ?$$

Use properties of the cross product and our earlier calculations for the standard basis vectors

$$\langle a_1, a_2, a_3\rangle \times \langle b_1, b_2, b_3\rangle$$

$$= (a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}) \times (b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k})$$

$$= (a_1\hat{\imath}) \times (b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}) \leftarrow$$

$$+ (a_2\hat{\jmath}) \times (b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k})$$

$$+ (a_3\hat{k}) \times (b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k})$$

$$(a_1\hat{\imath}) \times (b_1\hat{\imath}) = \vec{0}$$
$$+ (a_1\hat{\imath}) \times (b_2\hat{\jmath}) = a_1 b_2 \hat{k}$$
$$+ (a_1\hat{\imath}) \times (b_3\hat{k}) = -a_1 b_3 \hat{\jmath}$$

$$= \cdots$$

$$= (a_2 b_3 - a_3 b_2)\hat{\imath} + (a_3 b_1 - a_1 b_3)\hat{\jmath} + (a_1 b_2 - a_2 b_1)\hat{k}$$

$$= \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1\rangle$$

Ways to calculate the cross product

Determinants

2×2 determinant looks like $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

example: $\begin{vmatrix} 1 & 2 \\ -3 & 4 \end{vmatrix} = (1)(4) - (2)(-3) = 10$

3×3 determinant looks like

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

3×3 determinant looks like

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Example: $\begin{vmatrix} 1 & 5 & 2 \\ 2 & 4 & 1 \\ 0 & 3 & 1 \end{vmatrix} = 1 \begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} - 5 \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + 2 \begin{vmatrix} 2 & 4 \\ 0 & 3 \end{vmatrix}$

$$= 1 - 10 + 12 = 3$$

$$\langle a_1, a_2, a_3 \rangle \times \langle b_1, b_2, b_3 \rangle$$

$$= (a_2 b_3 - a_3 b_2)\hat{\imath} + (a_3 b_1 - a_1 b_3)\hat{\jmath} + (a_1 b_2 - a_2 b_1)\hat{k}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{\imath} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{k}$$

$$\boxed{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}}$$

So, $\langle a_1, a_2, a_3 \rangle \times \langle b_1, b_2, b_3 \rangle = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Example Find a unit vector perpendicular to $\langle 1,1,0\rangle$ and $\langle 0,2,3\rangle$.

$$\langle 1,1,0\rangle \times \langle 0,2,3\rangle = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 0 \\ 0 & 2 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix}\hat{\imath} - \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix}\hat{\jmath} + \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}\hat{k}$$

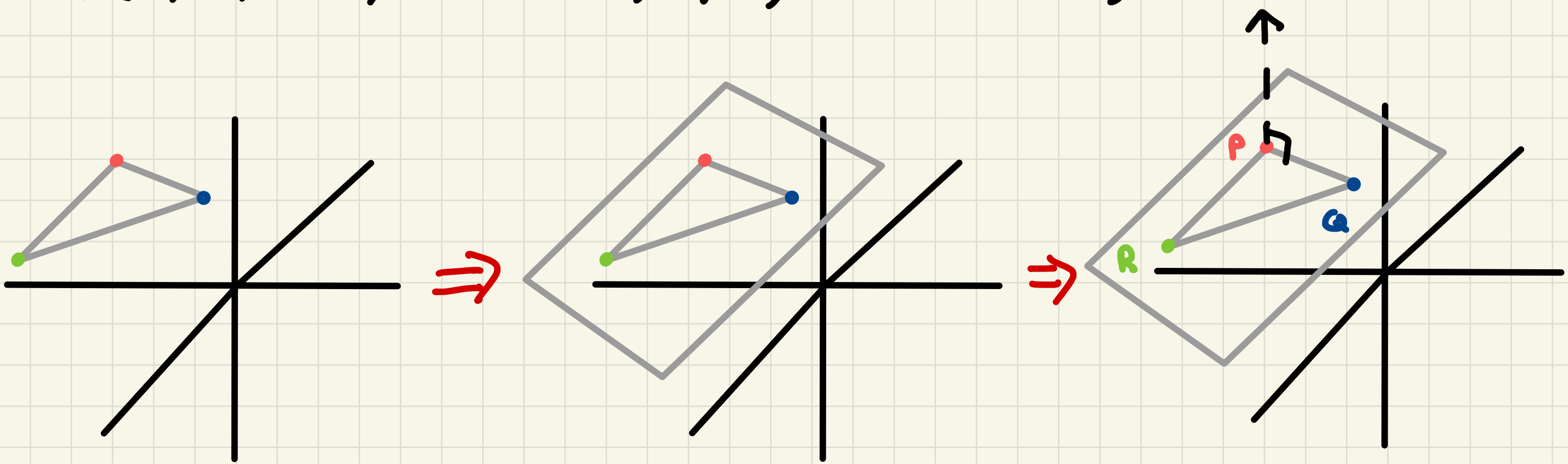
$$= 3\hat{\imath} - 3\hat{\jmath} + 2\hat{k} = \vec{v}$$

$$|\vec{v}| = \sqrt{3^2 + (-3)^2 + 2^2} = \sqrt{22}$$

$$\frac{\vec{v}}{|\vec{v}|} = \left\langle \frac{3}{\sqrt{22}}, \frac{-3}{\sqrt{22}}, \frac{2}{\sqrt{22}} \right\rangle$$ is one answer

other answer ?

Example Find a vector perpendicular to the plane that passes through the points $P(1,2,3)$, $Q(4,5,6)$, $R(2,1,3)$



take cross product $\overrightarrow{PQ} \times \overrightarrow{PR}$

Example (cont.)

$P(1,2,3)$, $Q(4,5,6)$, $R(2,1,3)$

$\vec{PQ} = \langle 3,3,3 \rangle$, $\vec{PR} = \langle 1,-1,0 \rangle$

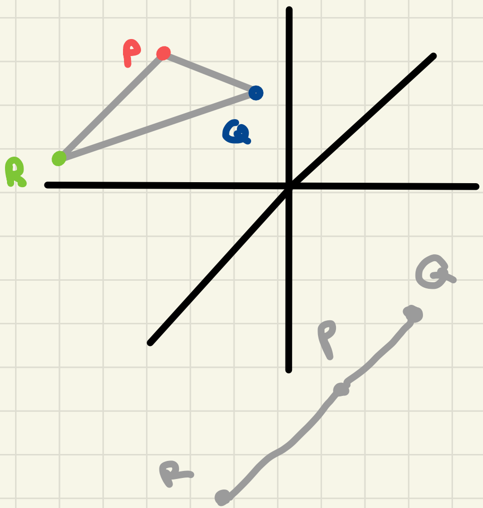
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 3 & 3 \\ 1 & -1 & 0 \end{vmatrix} = \hat{i}\begin{vmatrix} 3 & 3 \\ -1 & 0 \end{vmatrix} - \hat{j}\begin{vmatrix} 3 & 3 \\ 1 & 0 \end{vmatrix} + \hat{k}\begin{vmatrix} 3 & 3 \\ 1 & -1 \end{vmatrix}$$

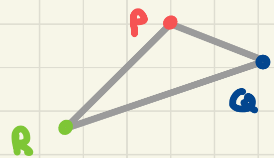
$$= 3\hat{i} + 3\hat{j} - 6\hat{k}$$

Example

Find the area of the triangle with vertices

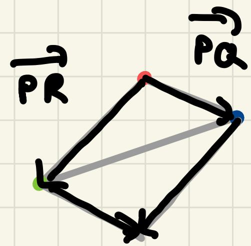
P(1,2,3), Q(4,5,6), R(2,1,3)





$$\frac{|\vec{PQ}\times\vec{PR}|}{2}$$

$$|3\hat{i}+3\hat{j}-6\hat{k}|/2$$



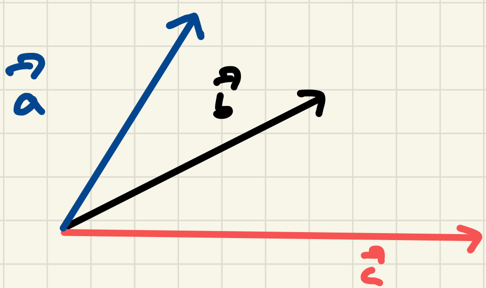
parallelogram

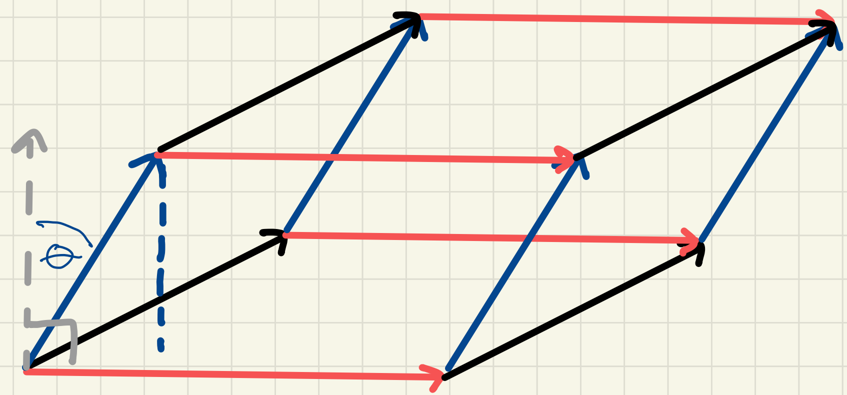
$|\vec{PQ}\times\vec{PR}|$ = area of parallelogram

Intermission

Let $\vec{a}, \vec{b}, \vec{c}$ be vectors

Note that the three vectors determine a parallelepiped



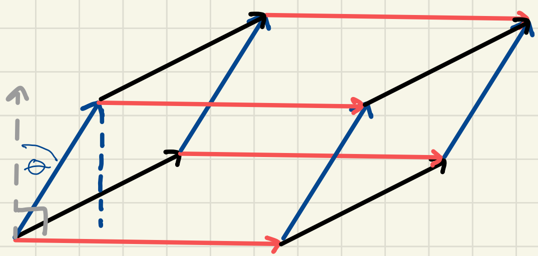


What is the volume? $|\vec{b} \times \vec{c}| \, \big| |\vec{a}| \cos(\theta) \big|$

$$= |\vec{a} \cdot (\vec{b} \times \vec{c})|$$

$\vec{a} \cdot (\vec{b} \times \vec{c})$ is called the scalar triple product

The volume of the parallelepiped determined by $\vec{a}$, $\vec{b}$, $\vec{c}$ is the absolute value of the scalar triple product: $|\vec{a} \cdot (\vec{b} \times \vec{c})|$



- $\vec{a}$
- $\vec{b}$
- $\vec{c}$

Note: $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{c} \cdot (\vec{a} \times \vec{b})$

$= (\vec{a} \times \vec{b}) \cdot \vec{c}$

Proof: tilt head or work it out using the formula with components

How to compute $\vec{a} \cdot (\vec{b} \times \vec{c})$?

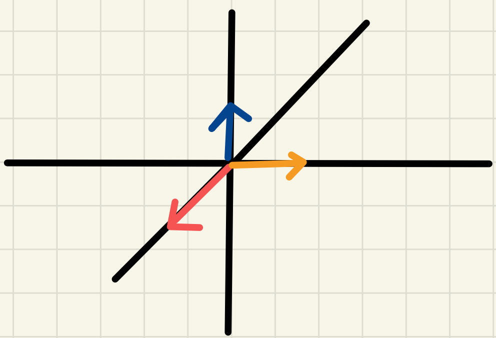
$$= \vec{a} \cdot \left(\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \hat{\imath} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \hat{k} \right)$$

$$= \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} a_1 - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} a_2 + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} a_3$$

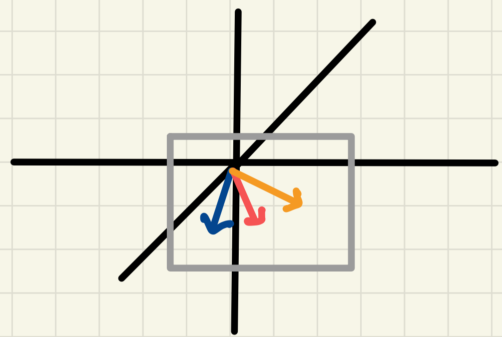
$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Application: are the vectors $\langle 1,1,2 \rangle$, $\langle 2,4,7 \rangle$, $\langle 5,9,16 \rangle$ coplanar

(do they all lie on a single plane)



not coplanar



coplanar

how to tell? no volume because the parallelepiped would be flat!

Application: are the vectors

$\langle 1,1,2\rangle$, $\langle 2,4,7\rangle$, $\langle 5,9,16\rangle$ coplanar

(do they all lie on a single plane)

$$\begin{vmatrix} 1 & 1 & 2 \\ 2 & 4 & 7 \\ 5 & 9 & 16 \end{vmatrix} = \begin{vmatrix} 4 & 7 \\ 9 & 16 \end{vmatrix} 1 - \begin{vmatrix} 2 & 7 \\ 5 & 16 \end{vmatrix} 1 + \begin{vmatrix} 2 & 4 \\ 5 & 9 \end{vmatrix} 2$$

$$= (64-63)1 - (32-35)1 + (18-20)2$$

$$= 1 + 3 - 4$$

$$= 0$$, coplanar