

Cross product

Last time: dot product  $\vec{a} \cdot \vec{b} \in \mathbb{R}$

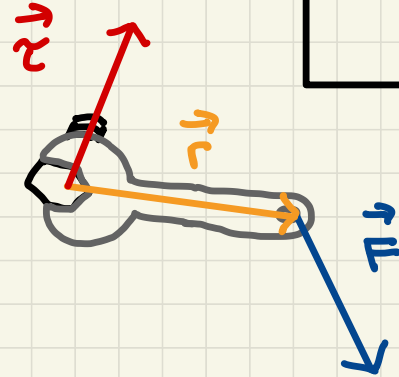
(informally, measures the extent to which  $\vec{a}$  and  $\vec{b}$  point in the same direction)

Today: cross product  $\vec{a} \times \vec{b}$

- is a vector perpendicular to  $\vec{a}$  and  $\vec{b}$
- natural geometric interpretation

# Torque

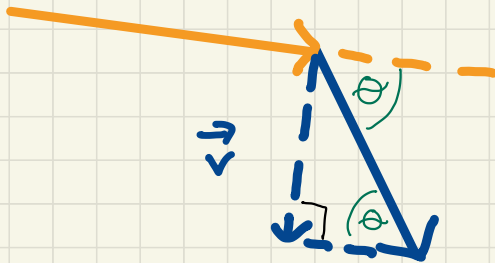
Note:  $\vec{r} \perp \vec{\tau}$   
 $\vec{r} \perp \vec{F}$

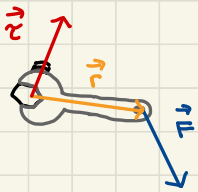


$|\vec{\tau}|$  depends on  $|\vec{r}|$   
 $|\vec{F}|$

AND angle  $\theta$  between  $\vec{r}$  and  $\vec{F}$

the length  $|\vec{\tau}| = |\vec{F}| \sin(\theta)$



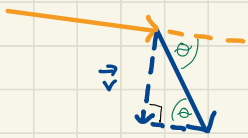


$|\vec{\tau}|$  depends on

on  $|\vec{r}|$   
 $|\vec{F}|$

AND angle  $\theta$  between  $\vec{r}$  and  $\vec{F}$

the length  $|\vec{v}| = |\vec{F}| \sin(\theta)$



So, the magnitude of our torque is

$$|\vec{\tau}| = |\vec{r}| |\vec{F}| \sin(\theta)$$

What about the direction?

Let  $\vec{n}$  be a unit vector in the direction in which the right-handed bolt moves

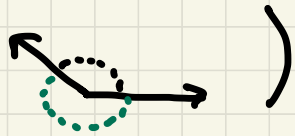
$$\text{Then } \vec{r} \times \vec{F} = \underbrace{(|\vec{r}| |\vec{F}| \sin(\theta))}_{\text{magnitude}} \underbrace{\vec{n}}_{\text{direction}}$$

This is the cross product

## General definition of the cross product

If  $\vec{a}, \vec{b}$  are vectors in  $\mathbb{R}^3$ ,

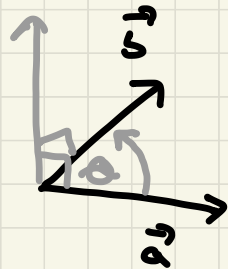
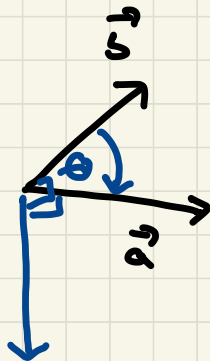
then  $\vec{a} \times \vec{b} = (|\vec{a}| |\vec{b}| \sin(\theta)) \vec{n}$ ,

where  $\theta$  is the angle between  $\vec{a}$  and  $\vec{b}$   
with  $\theta \in [0, \pi]$  (recall )

and  $\vec{n}$  is a unit vector perpendicular to both  $\vec{a}$  and  $\vec{b}$  and whose direction is given by the right-hand rule: curl your fingers through the angle from  $\vec{a}$  to  $\vec{b}$  (NOT  $\vec{b}$  to  $\vec{a}$ ), then your thumb points in the direction of  $\vec{n}$

$$\vec{a} \times \vec{b} = (|\vec{a}| |\vec{b}| \sin(\theta)) \vec{n}$$

Immediate properties :  $\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}$



Again,  $\vec{a} \times \vec{b} \perp \vec{a}$   
 $\vec{a} \times \vec{b} \perp \vec{b}$

If  $\vec{a}, \vec{b} \neq \vec{0}$ ,

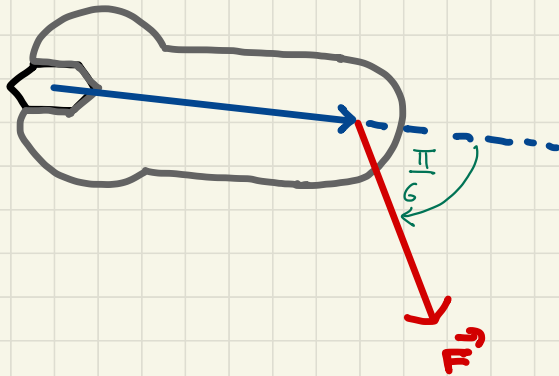
then

$\vec{a}$  and  $\vec{b}$  are parallel  $\iff \vec{a} \times \vec{b} = \vec{0}$



## Example

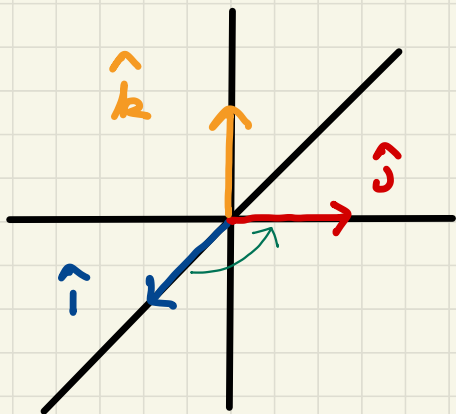
A bolt is tightened by applying a 100 N force to a .1 m wrench as below



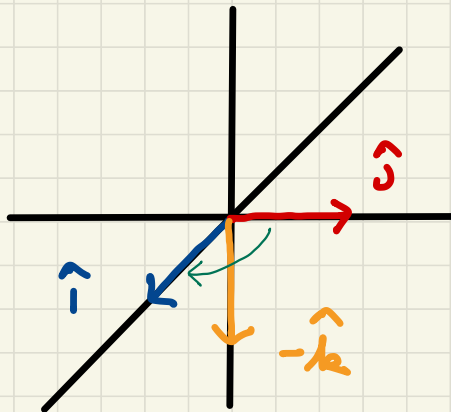
Find the magnitude of the torque

# Standard basis vectors

$$\hat{i} \times \hat{j} = \hat{k}$$



$$\hat{j} \times \hat{i} = -\hat{k}$$



Exercise :  $\hat{i} \times \hat{k} = -\hat{j}$

$$\hat{k} \times \hat{i} = \hat{j}$$

$$\hat{j} \times \hat{k} = \hat{i}$$

$$\hat{k} \times \hat{j} = -\hat{i}$$



## Some warnings

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

$$(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c}) \quad \text{in general}$$

$$\text{example: } (\hat{i} \times \hat{i}) \times \hat{j} = \vec{0} \times \hat{j} = \vec{0}$$

$$\hat{i} \times (\hat{i} \times \hat{j}) = \hat{i} \times \hat{k} = -\hat{j} \neq \vec{0}$$

in other words, not associative

## Properties of the cross product

You will **not** be responsible for the proofs, but we need these properties

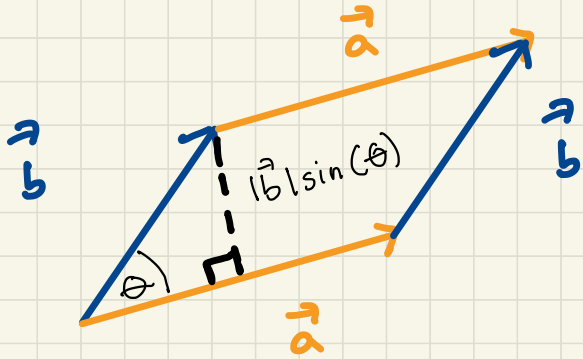
$$1. \vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

$$2. (c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$$

$$3. \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

$$4. (\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$$

## Geometric interpretation



Area of the parallelogram is  $|\vec{a}||\vec{b}|\sin(\theta)$ , which is the magnitude of the cross product  $\vec{a} \times \vec{b}$

In other words,  $|\vec{a} \times \vec{b}|$  is the area of the parallelogram (note  $|\vec{a} \times \vec{b}| = |\vec{b} \times \vec{a}|$ )

Intermission

## Components

$$\langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\langle a_1, a_2, a_3 \rangle \times \langle b_1, b_2, b_3 \rangle = ?$$

Rewrite  $\langle a_1, a_2, a_3 \rangle = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$

$$\langle b_1, b_2, b_3 \rangle = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$$

$$(a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \times (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}) = ?$$

Use properties of the cross product  
and our earlier calculations for the standard  
basis vectors

$$\langle a_1, a_2, a_3 \rangle \times \langle b_1, b_2, b_3 \rangle$$

$$= (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \times (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k})$$

$$= (a_1 \hat{i}) \times (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k})$$

$$\begin{aligned} \leftarrow \begin{aligned} (a_1 \hat{i}) \times (b_1 \hat{i}) &= \vec{0} \\ + (a_1 \hat{i}) \times (b_2 \hat{j}) &= a_1 b_2 \hat{k} \\ + (a_1 \hat{i}) \times (b_3 \hat{k}) &= -a_1 b_3 \hat{j} \end{aligned} \end{aligned}$$

$$+ (a_2 \hat{j}) \times (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k})$$

$$+ (a_3 \hat{k}) \times (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k})$$

$$= \dots$$

$$= (a_2 b_3 - a_3 b_2) \hat{i} + (a_3 b_1 - a_1 b_3) \hat{j} + (a_1 b_2 - a_2 b_1) \hat{k}$$

$$= \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

# Ways to calculate the cross product

## Determinants

2x2 determinant looks like  $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

example:  $\begin{vmatrix} 1 & 2 \\ -3 & 4 \end{vmatrix} = (1)(4) - (2)(-3) = 10$

3x3 determinant looks like

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

3x3 determinant looks like

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Example:  $\begin{vmatrix} 1 & 5 & 2 \\ 2 & 4 & 1 \\ 0 & 3 & 1 \end{vmatrix} = 1 \begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} - 5 \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + 2 \begin{vmatrix} 2 & 4 \\ 0 & 3 \end{vmatrix}$

$$= 1 - 10 + 12 = 3$$



$$\langle a_1, a_2, a_3 \rangle \times \langle b_1, b_2, b_3 \rangle$$

$$= (a_2 b_3 - a_3 b_2) \hat{i} + (a_3 b_1 - a_1 b_3) \hat{j} + (a_1 b_2 - a_2 b_1) \hat{k}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{k}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$\text{So, } \langle a_1, a_2, a_3 \rangle \times \langle b_1, b_2, b_3 \rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example Find a unit vector perpendicular to  $\langle 1, 1, 0 \rangle$  and  $\langle 0, 2, 3 \rangle$ .

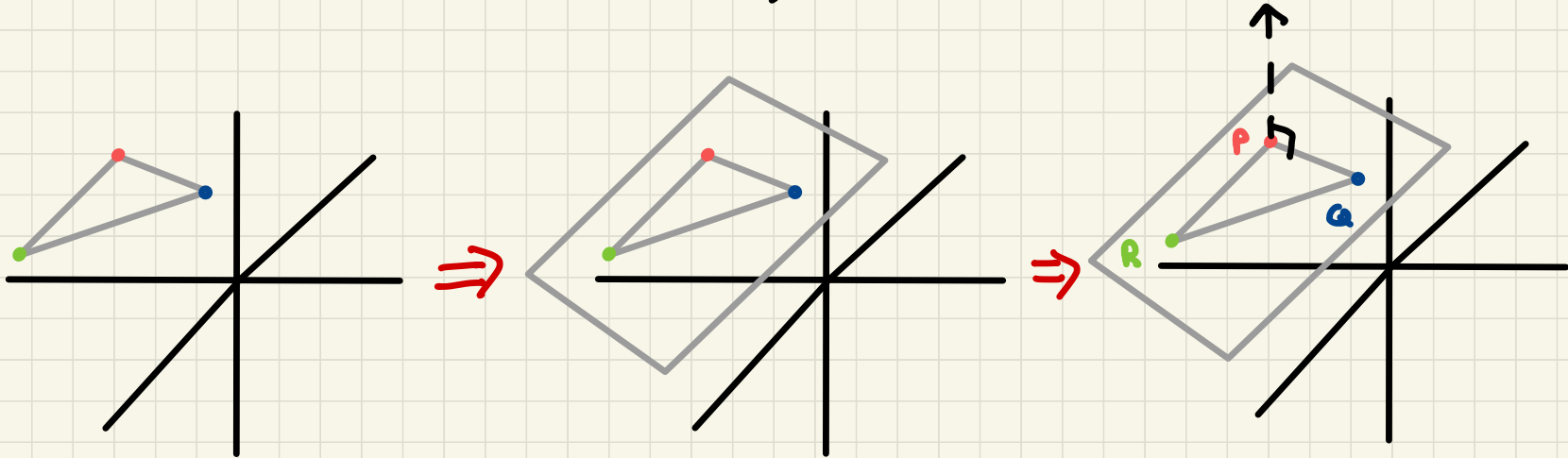
$$\begin{aligned}\langle 1, 1, 0 \rangle \times \langle 0, 2, 3 \rangle &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 0 & 2 & 3 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} \hat{i} - \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} \hat{j} + \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} \hat{k} \\ &= 3\hat{i} - 3\hat{j} + 2\hat{k} = \vec{v}\end{aligned}$$

$$|\vec{v}| = \sqrt{3^2 + (-3)^2 + 2^2} = \sqrt{20}$$

$$\frac{\vec{v}}{|\vec{v}|} = \left\langle \frac{3}{\sqrt{20}}, \frac{-3}{\sqrt{20}}, \frac{2}{\sqrt{20}} \right\rangle \text{ is one answer}$$

other answer?

Example Find a vector perpendicular to the plane that passes through the points  $P(1, 2, 3)$ ,  $Q(4, 5, 6)$ ,  $R(2, 1, 3)$



take cross product  $\vec{PQ} \times \vec{PR}$

## Example (cont.)

$$P(1, 2, 3), \quad Q(4, 5, 6), \quad R(2, 1, 3)$$

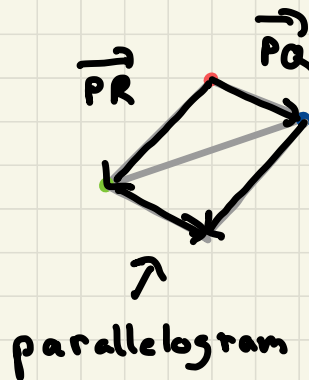
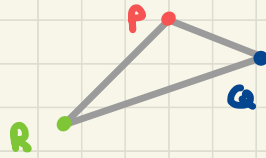
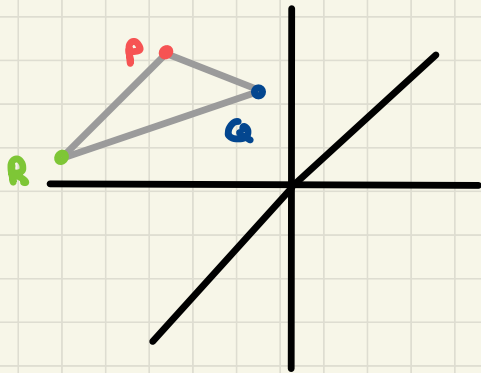
$$\vec{PQ} = \langle 3, 3, 3 \rangle, \quad \vec{PR} = \langle 1, -1, 0 \rangle$$

$$\begin{aligned} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 3 & 3 \\ 1 & -1 & 0 \end{vmatrix} &= \hat{i} \begin{vmatrix} 3 & 3 \\ -1 & 0 \end{vmatrix} - \hat{j} \begin{vmatrix} 3 & 3 \\ 1 & 0 \end{vmatrix} + \hat{k} \begin{vmatrix} 3 & 3 \\ 1 & -1 \end{vmatrix} \\ &= 3\hat{i} + 3\hat{j} - 6\hat{k} \end{aligned}$$

## Example

Find the area of the triangle with vertices

$P(1, 2, 3)$ ,  $Q(4, 5, 6)$ ,  $R(2, 1, 3)$

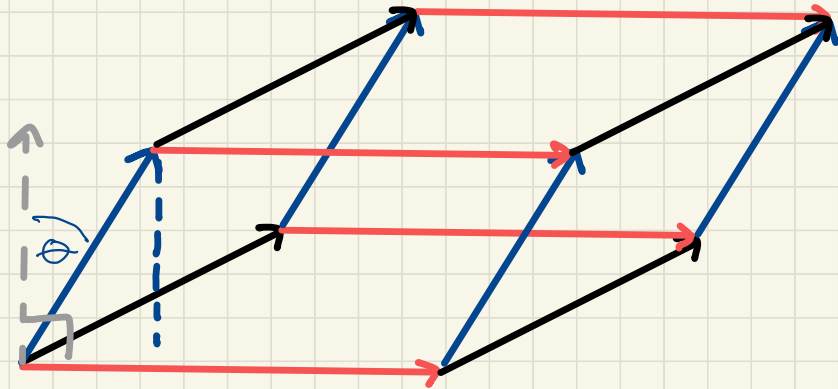
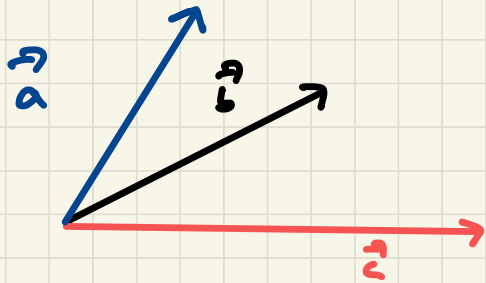


$$|\vec{PQ} \times \vec{PR}| = \text{area of parallelogram}$$

Intermission

Let  $\vec{a}, \vec{b}, \vec{c}$  be vectors

Note that the three vectors determine  
a parallelepiped

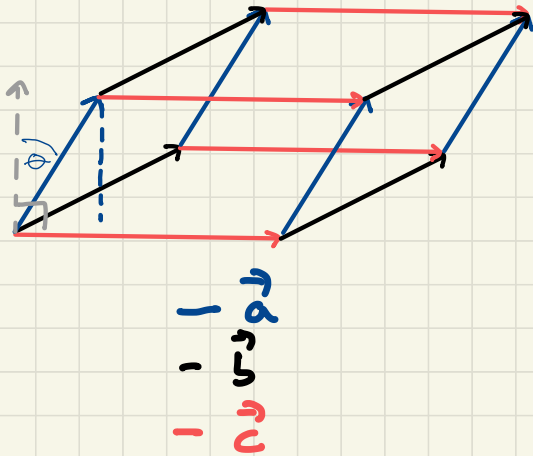


What is the volume?

$$|\vec{b} \times \vec{c}| |\vec{a}| \cos(\theta)$$
$$= \vec{a} \cdot (\vec{b} \times \vec{c})$$

$\vec{a} \cdot (\vec{b} \times \vec{c})$  is called the **scalar triple product**

The volume of the parallelepiped determined by  $\vec{a}, \vec{b}, \vec{c}$  is the absolute value of the scalar triple product:  $|\vec{a} \cdot (\vec{b} \times \vec{c})|$



Note:  $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{c} \cdot (\vec{a} \times \vec{b})$   
 $= (\vec{a} \times \vec{b}) \cdot \vec{c}$

Proof: tilt head or work it out using the formula with components



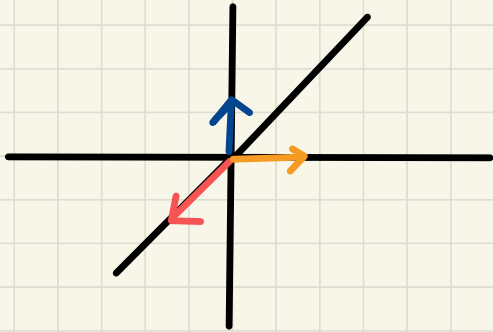
How to compute  $\vec{a} \cdot (\vec{b} \times \vec{c})$ ?

$$= \vec{a} \cdot \left( \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \hat{i} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \hat{j} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \hat{k} \right)$$

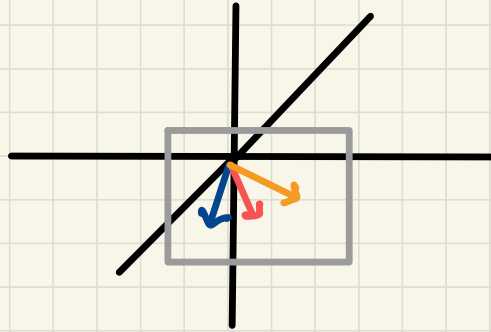
$$= \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} a_1 - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} a_2 + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} a_3$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Application : are the vectors  
 $\langle 1, 1, 2 \rangle$ ,  $\langle 2, 4, 7 \rangle$ ,  $\langle 5, 9, 16 \rangle$  coplanar  
(do they all lie on a single plane)



not coplanar



coplanar

how to tell? no volume because the  
parallelepiped would be flat!

Application : are the vectors  
 $\langle 1, 1, 2 \rangle$ ,  $\langle 2, 4, 7 \rangle$ ,  $\langle 5, 9, 16 \rangle$  coplanar  
(do they all lie on a single plane)

$$\begin{vmatrix} 1 & 1 & 2 \\ 2 & 4 & 7 \\ 5 & 9 & 16 \end{vmatrix} = \begin{vmatrix} 4 & 7 \\ 9 & 16 \end{vmatrix} 1 - \begin{vmatrix} 2 & 7 \\ 5 & 16 \end{vmatrix} 1 + \begin{vmatrix} 2 & 4 \\ 5 & 9 \end{vmatrix} 2$$

$$= (64 - 63)1 - (32 - 35)1 + (18 - 20)2$$

$$= 1 + 3 - 4$$

$$= 0, \quad \text{coplanar}$$