

Derivatives and integrals of vector functions

Last time : domain of vector functions

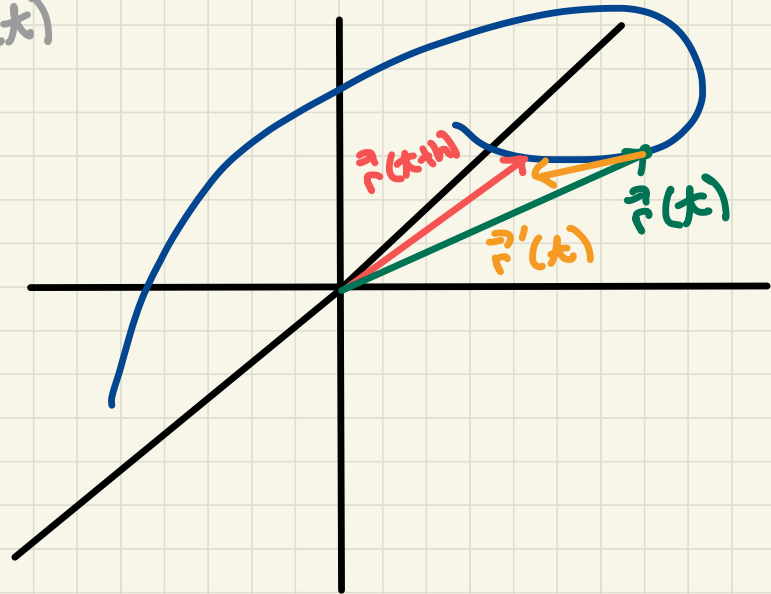
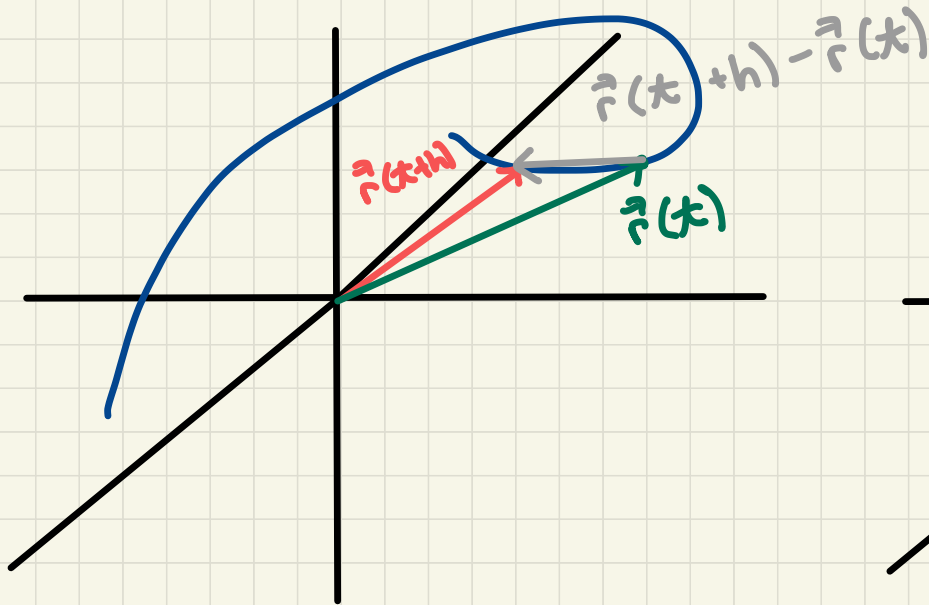
limit of vector functions

continuity of vector functions

Moral : work with the component functions

Let $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

then $\frac{d\vec{r}}{dt} = \vec{r}'(t) = \lim_{h \rightarrow 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$



If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, where

f, g, h are differentiable, then

$$\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

Pf $\vec{r}'(t) = \lim_{s \rightarrow 0} \frac{\vec{r}(t+s) - \vec{r}(t)}{s}$

$$= \lim_{s \rightarrow 0} \left\langle \frac{f(t+s) - f(t)}{s}, \frac{g(t+s) - g(t)}{s}, \frac{h(t+s) - h(t)}{s} \right\rangle$$

$$= \left\langle \lim_{s \rightarrow 0} \frac{f(t+s) - f(t)}{s}, \lim_{s \rightarrow 0} \frac{g(t+s) - g(t)}{s}, \lim_{s \rightarrow 0} \frac{h(t+s) - h(t)}{s} \right\rangle$$

Given $\vec{r}(t)$, we can try to compute

- the derivative $\vec{r}'(t)$ (the tangent vector)
- the unit tangent vector $\frac{\vec{r}'(t)}{|\vec{r}'(t)|}$
- the tangent line to $\vec{r}(t)$ at a point P

Example: find the derivative of

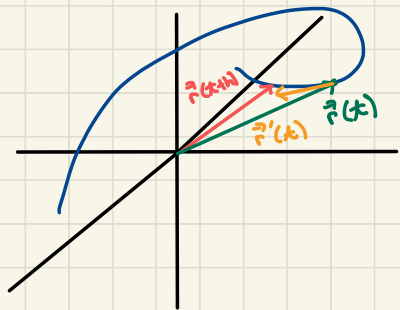
$$\vec{r}(t) = \left\langle \frac{t+1}{t^2+1}, \cos(t^3), t e^{-t} \right\rangle$$

find the unit tangent vector as well as an equation for the tangent line at time $t = 0$

Ans:

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$$\text{Ans: } \vec{r}'(t) = \left\langle \frac{t^2+1-2t(t+1)}{(t^2+1)^2}, -\sin(t^3)(3t^2), e^{-t}-te^{-t} \right\rangle$$

$$\vec{r}'(0) = \langle 1, 0, 1 \rangle, \quad \frac{\vec{r}'(0)}{|\vec{r}'(0)|} = \frac{\langle 1, 0, 1 \rangle}{\sqrt{2}}$$

$$\vec{r}(0) = \langle 1, 1, 0 \rangle \quad \text{so tangent line is} \quad \begin{array}{l} x=1+t \\ y=1 \\ z=t \end{array}$$

Interpretation :

$\vec{r}(t)$ position

$\vec{r}'(t)$ velocity

$\vec{r}''(t)$ acceleration

more on this later

Rules for differentiation

$$1. \frac{d}{dt} [\vec{u}(t) + \vec{v}(t)] = \vec{u}'(t) + \vec{v}'(t)$$

$$2. \frac{d}{dt} [c\vec{u}(t)] = c\vec{u}'(t)$$

$$3. \frac{d}{dt} [f(t)\vec{u}(t)] = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$$

$$4. \frac{d}{dt} [\vec{u}(t) \cdot \vec{v}(t)] = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$$

$$5. \frac{d}{dt} [\vec{u}(t) \times \vec{v}(t)] = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$$

$$6. \frac{d}{dt} [\vec{u}(f(t))] = f'(t) \vec{u}'(f(t))$$

Example: Show that if $|\vec{r}(t)| = c$ for all t , then $\vec{r}'(t)$ is orthogonal to $\vec{r}(t)$ for all t

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Ans: $|\vec{r}(t)| = c \Rightarrow |\vec{r}(t)|^2 = c^2$

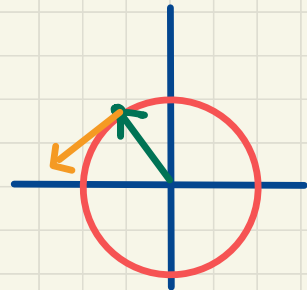
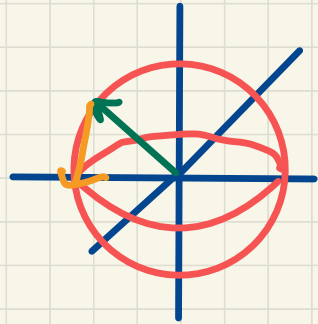
$$|\vec{r}(t)|^2 = \vec{r}(t) \cdot \vec{r}(t)$$

$$\frac{d}{dt} [\vec{r}(t) \cdot \vec{r}(t)] = \frac{d}{dt} [c^2] = 0$$

$$\vec{r}'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}'(t) = 0$$

$$2 \vec{r}'(t) \cdot \vec{r}(t) = 0$$

$$\vec{r}'(t) \cdot \vec{r}(t) = 0$$



Integrals of vector functions

$$\text{if } \vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

$$\text{then } \int_a^b \vec{r}(t) dt = \left\langle \int_a^b f(t) dt, \int_a^b g(t) dt, \int_a^b h(t) dt \right\rangle$$

by the fundamental theorem of calculus,

$$\text{if } \vec{R}'(t) = \vec{r}(t),$$

$$\text{then } \int_a^b \vec{r}(t) dt = \vec{R}(b) - \vec{R}(a)$$

Example: $\vec{r}(t) = \langle e^{-t}, \sqrt{t}, \cos(\pi t) \rangle$

compute $\int_0^1 \vec{r}(t) dt$

Ans:

$$\int \vec{r}(t) dt = \left\langle -e^{-t}, \frac{2}{3}t^{3/2}, \frac{\sin(\pi t)}{\pi} \right\rangle + \vec{C}$$

$$\begin{aligned} \int_0^1 \vec{r}(t) dt &= \left\langle -e^{-t} \Big|_0^1, \frac{2}{3}t^{3/2} \Big|_0^1, \frac{\sin(\pi t)}{\pi} \Big|_0^1 \right\rangle \\ &= \left\langle -e^{-1} + 1, \frac{2}{3}, 0 \right\rangle \end{aligned}$$