Derivatives and integrals of vector functions

Last time: domain of vector functions

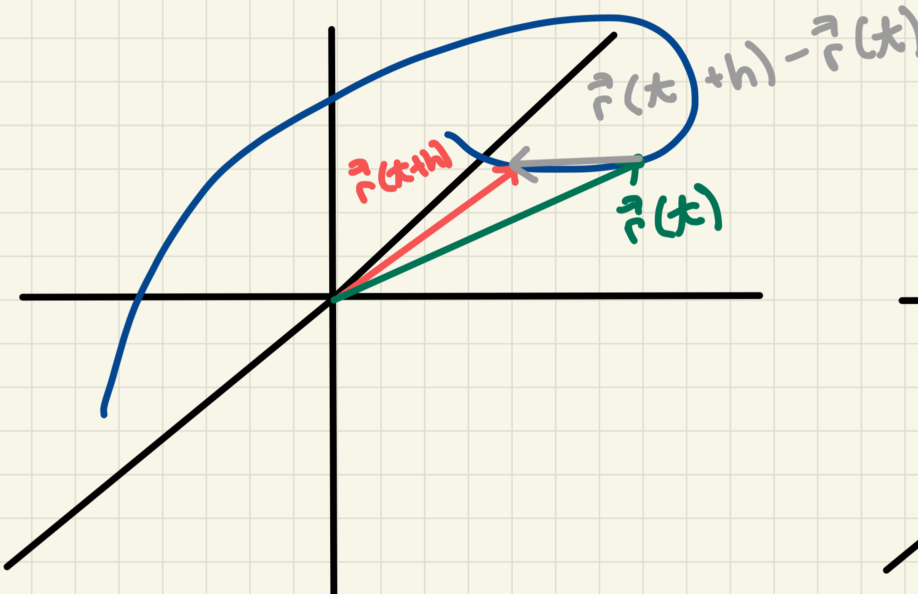
limit of vector functions

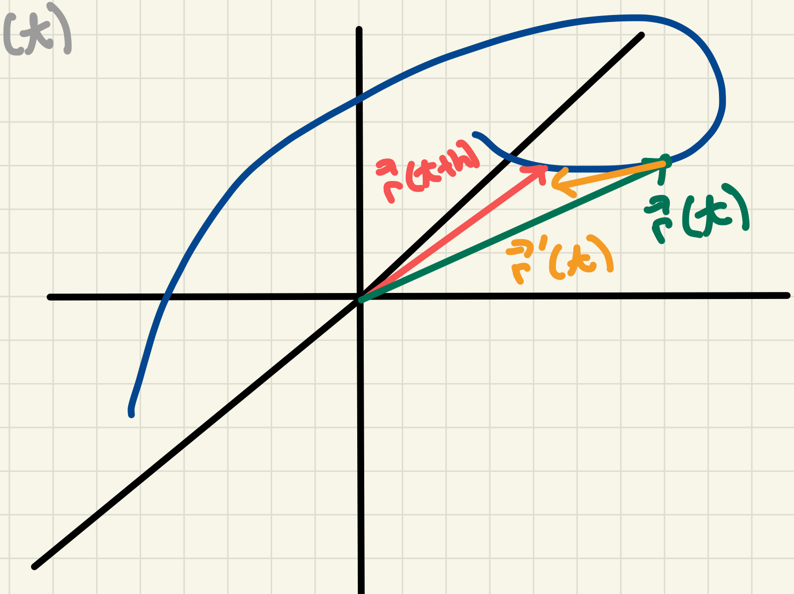
continuity of vector functions

Moral: work with the component functions

Let $\vec{r}(t) = \langle f(t),\ g(t),\ h(t) \rangle$

then $\dfrac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim\limits_{h \to 0} \dfrac{\vec{r}(t+h) - \vec{r}(t)}{h}$





If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, where

f, g, h are differentiable, then

$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

Pf $\quad \vec{r}\,'(t) = \lim_{s \to 0} \dfrac{\vec{r}(t+s) - \vec{r}(t)}{s}$

$$= \lim_{s \to 0} \left\langle \frac{f(t+s) - f(t)}{s}, \frac{g(t+s) - g(t)}{s}, \frac{h(t+s) - h(t)}{s} \right\rangle$$

$$= \left\langle \lim_{s \to 0} \frac{f(t+s) - f(t)}{s}, \lim_{s \to 0} \frac{g(t+s) - g(t)}{s}, \lim_{s \to 0} \frac{h(t+s) - h(t)}{s} \right\rangle$$

Given $\vec{r}(t)$, we can try to compute

- the derivative $\vec{r}\,'(t)$ (the tangent vector)
- the unit tangent vector $\dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$
- the tangent line to $\vec{r}(t)$ at a point P

Example: Find the derivative of

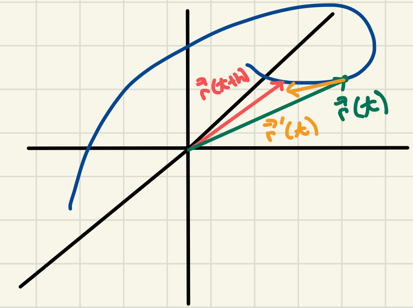
$$\vec{r}(t) = \left\langle \frac{t+1}{t^2+1} , \cos(t^3) , t e^{-t} \right\rangle$$

Find the unit tangent vector as well as an equation for the tangent line at time $t = 0$

Ans:

Example: Find the derivative of

$$\vec{r}(t) = \left\langle \frac{t+1}{t^2+1},\ \cos(t^3),\ te^{-t} \right\rangle$$



Find the unit tangent vector as well as an equation for the tangent line at time $t = 0$

Ans: $\vec{r}\,'(t) = \left\langle \frac{t^2+1-2t(t+1)}{(t^2+1)^2},\ -\sin(t^3)\,(3t^2),\ e^{-t} - te^{-t} \right\rangle$

$\vec{r}\,'(0) = \langle 1, 0, 1 \rangle$, $\quad \frac{\vec{r}\,'(0)}{|\vec{r}\,'(0)|} = \frac{\langle 1, 0, 1 \rangle}{\sqrt{2}}$

$\vec{r}(0) = \langle 1, 1, 0 \rangle$ so tangent line is
$x = 1 + t$
$y = 1$
$z = t$

Interpretation :

$\vec{r}(t)$ position

$\vec{r}\,'(t)$ velocity

$\vec{r}\,''(t)$ acceleration

more on this later

Rules for differentiation

1. $\frac{d}{dt}[\vec{u}(t)+\vec{v}(t)] = \vec{u}'(t) + \vec{v}'(t)$

2. $\frac{d}{dt}[c\vec{u}(t)] = c\vec{u}'(t)$

3. $\frac{d}{dt}[f(t)\vec{u}(t)] = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$

4. $\frac{d}{dt}[\vec{u}(t)\cdot\vec{v}(t)] = \vec{u}'(t)\vec{v}(t) + \vec{u}(t)\vec{v}'(t)$

5. $\frac{d}{dt}[\vec{u}(t)\times\vec{v}(t)] = \vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t)$

6. $\frac{d}{dt}[\vec{u}(f(t))] = f'(t)\,\vec{u}'(f(t))$

Example: Show that if $|r(t)| = c$ for all t, then $\vec{r}\,'(t)$ is orthogonal to $\vec{r}(t)$ for all t

Ans:

Example: Show that if $|r(t)| = c$ for all t, then $\vec{r}\,'(t)$ is orthogonal to $\vec{r}(t)$ for all t

Ans: $|\vec{r}(t)| = c \Rightarrow |\vec{r}(t)|^2 = c^2$

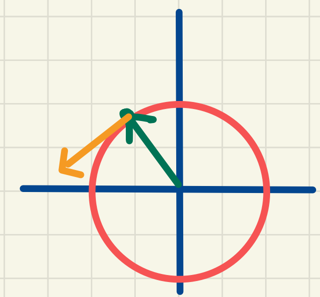
$$|\vec{r}(t)|^2 = \vec{r}(t) \cdot \vec{r}(t)$$

$$\frac{d}{dt}\left[\vec{r}(t) \cdot \vec{r}(t)\right] = \frac{d}{dt}\left[c^2\right] = 0$$

$$\vec{r}\,'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}\,'(t) = 0$$

$$2\,\vec{r}\,'(t) \cdot \vec{r}(t) = 0$$

$$\vec{r}\,'(t) \cdot \vec{r}(t) = 0$$



Integrals of vector functions

if $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

then $\int_a^b \vec{r}(t)\,dt = \langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \rangle$

by the fundamental theorem of calculus,

if $\vec{R}'(t) = \vec{r}(t)$,

then $\int_a^b \vec{r}(t)\,dt = \vec{R}(b) - \vec{R}(a)$

Example: $\vec{r}(t) = \left\langle e^{-t}, \sqrt{t}, \cos(\pi t) \right\rangle$

compute $\displaystyle\int_0^1 \vec{r}(t)\,dt$

Ans:

$$\int \vec{r}(t)\,dt = \left\langle -e^{-t}, \frac{2}{3}t^{3/2}, \frac{\sin(\pi t)}{\pi} \right\rangle + \vec{C}$$

$$\int_a^b \vec{r}(t)\,dt = \left\langle -e^{-t}\Big|_0^1, \frac{2}{3}t^{3/2}\Big|_0^1, \frac{\sin(\pi t)}{\pi}\Big|_0^1 \right\rangle$$

$$= \left\langle -e^{-1}+1, \frac{2}{3}, 0 \right\rangle$$