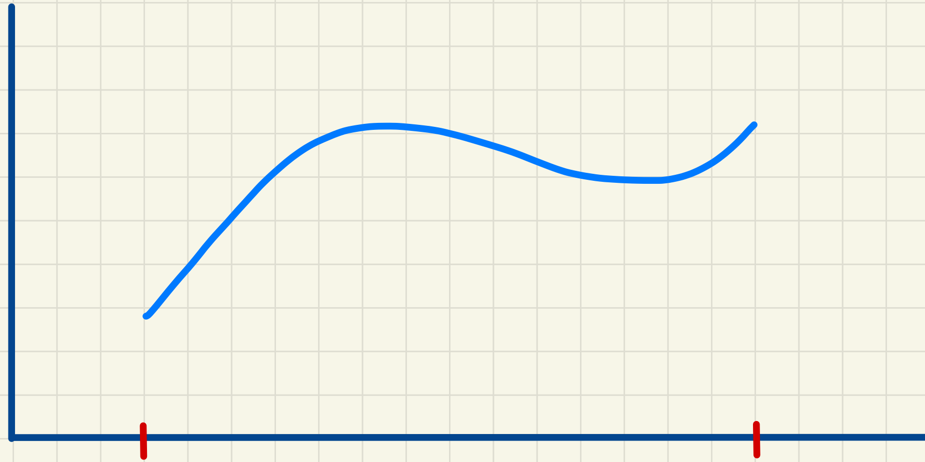
Double integrals over rectangles

Single integral over an interval:

let $f(x)$ be a function defined on an interval $[a,b]$

area under curve?



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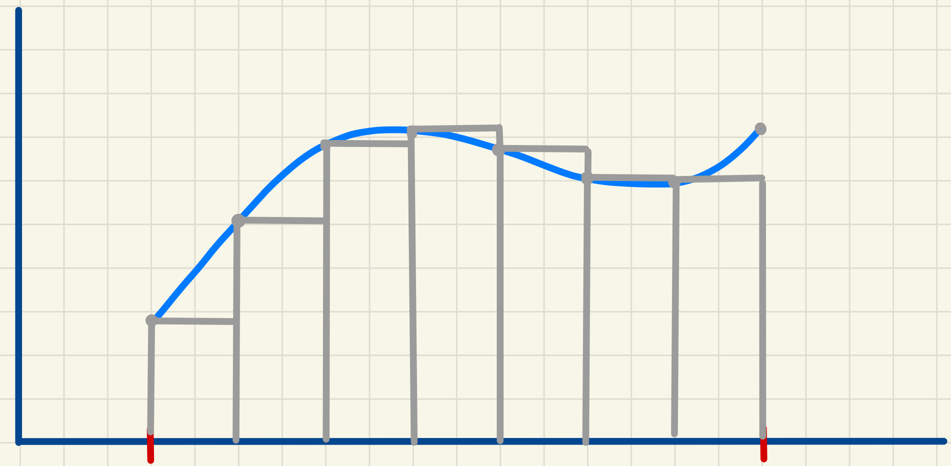

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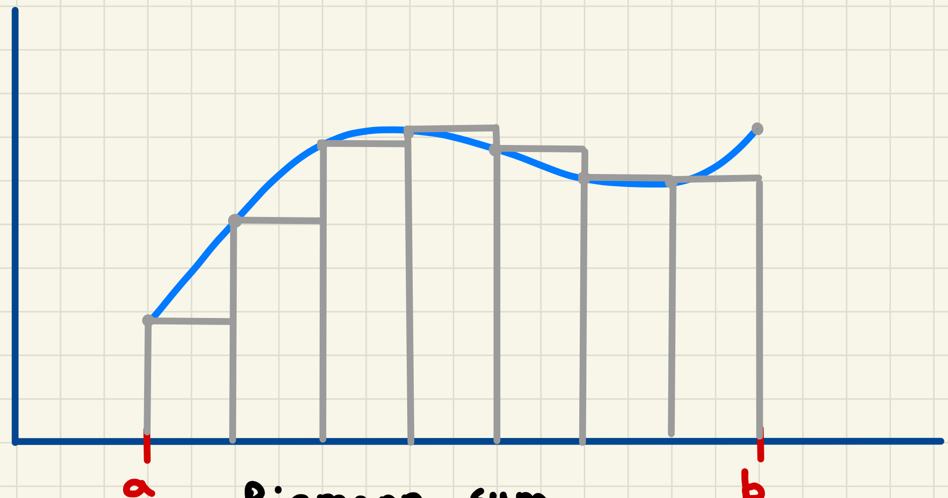


Riemann sum

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

sample point

What if $f(x,y)$ is a function of TWO variables?

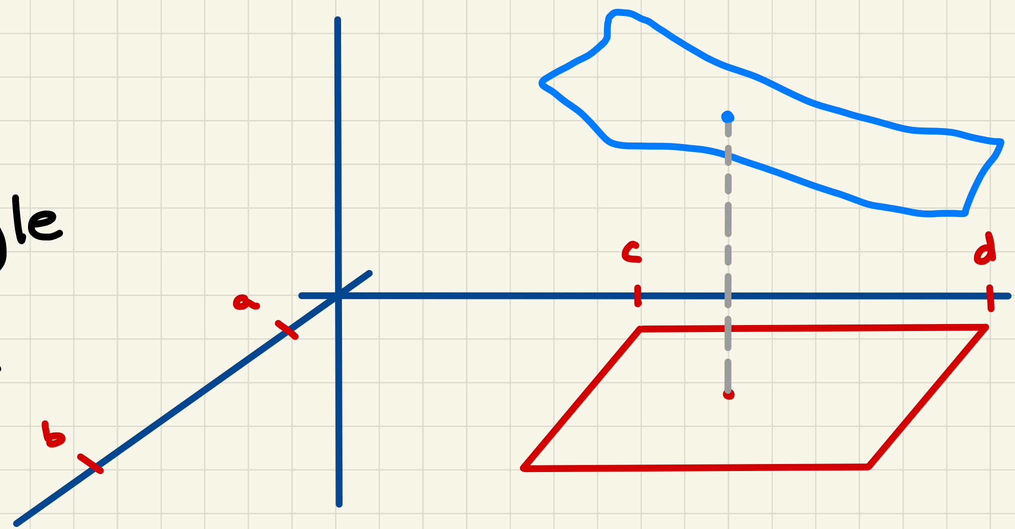
Instead of an interval $[a,b]$,

we have a rectangle $[a,b] \times [c,d]$

volume of solid

above the rectangle

and below $f(x,y)$?



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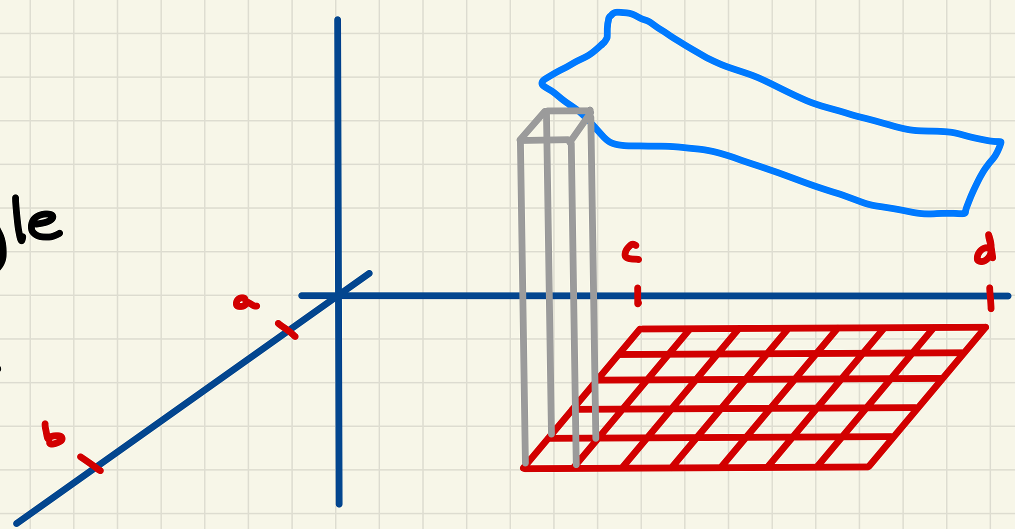
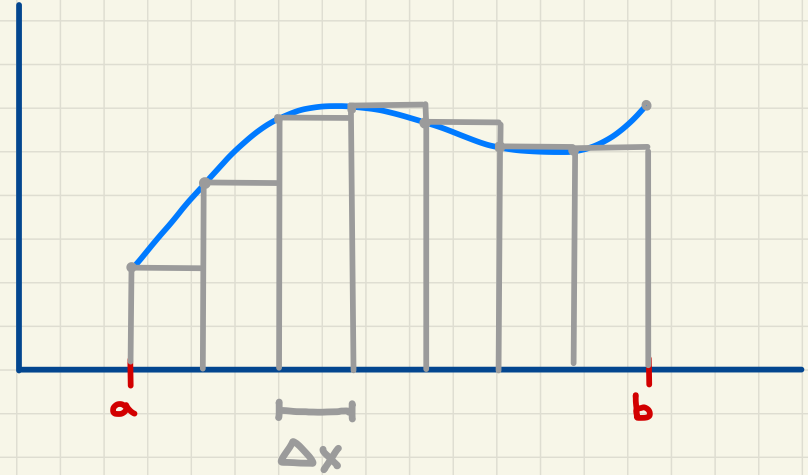
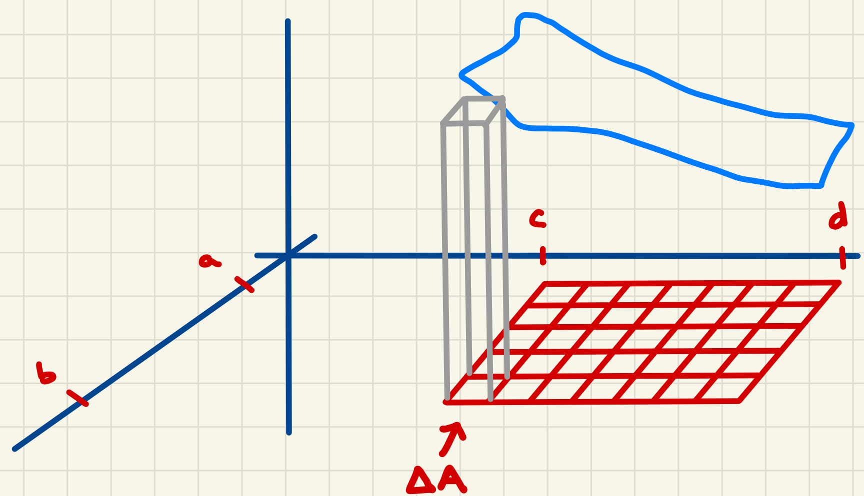


$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \overbrace{\sum_{i=1}^{n} f(x_i^*)\,\Delta x}^{\text{Riemann sum}}$$



$$V \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{i,j}^*, y_{i,j}^*)\,\Delta A$$

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{i,j}^*, y_{i,j}^*)\,\Delta A$$



The double integral of f over the rectangle R is

double Riemann sum

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{i,j}^*, y_{i,j}^*)\,\Delta A$$

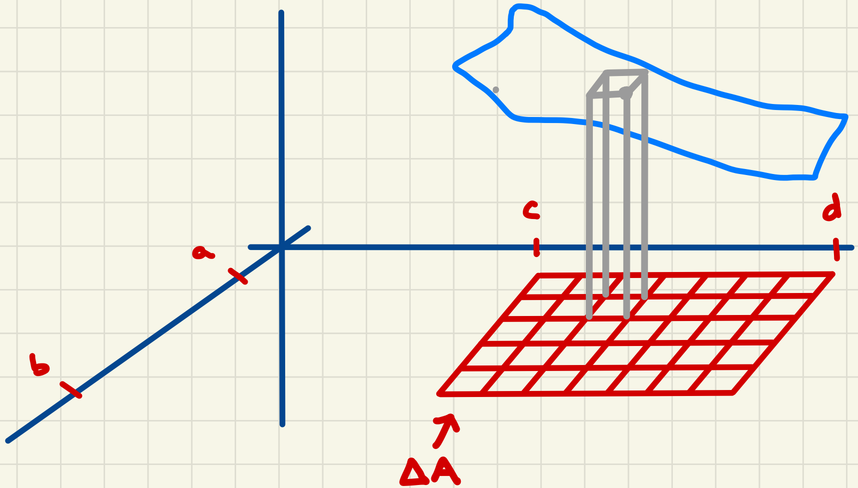
sample point

if this limit exists, in which case we call f INTEGRABLE

In the definition of

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{i,j}^*, y_{i,j}^*)\,\Delta A$$

we can choose $(x_{i,j}^*, y_{i,j}^*) = (x_{i,j}, y_{i,j})$ to be the upper right-hand corner of the sub-rectangle $R_{i,j}$ (analogue of a right Riemann sum)

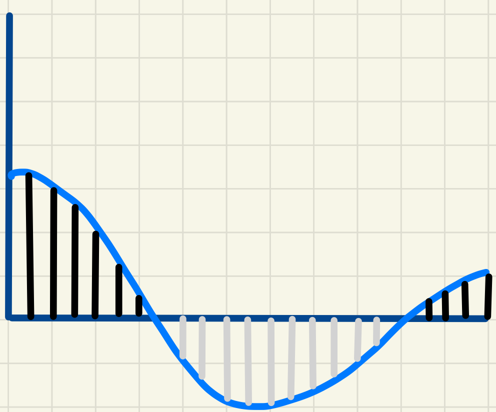


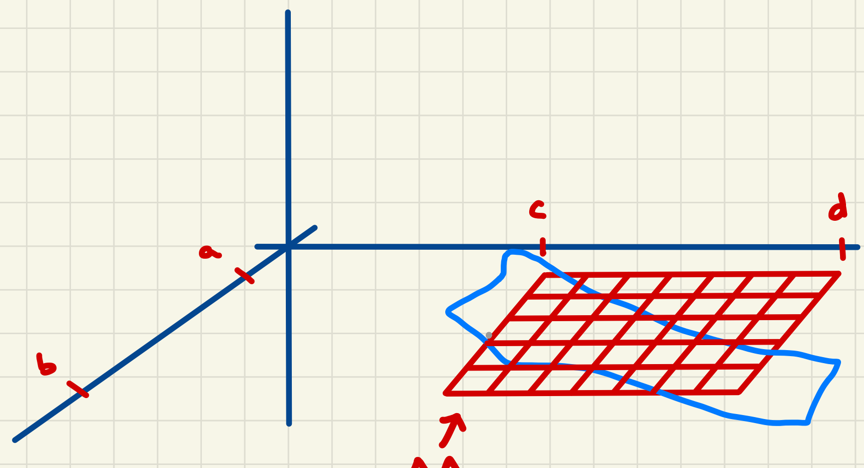
Then our formula becomes

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{i,j}, y_{i,j})\,\Delta A$$

If $f(x,y) \geq 0$ on R, then we can interpret $\iint_R f(x,y)\, dA$ as a volume

In general, we are computing signed volume





Example: estimate the volume of the solid that lies above the square $R = [1,3] \times [2,4]$ and below the function $f(x,y) = e^{x+y^2}$ by dividing R into 4 equal squares and choosing the sample point to be the upper right-corner

Example : $R = [1,3] \times [2,4]$

$f(x,y) = e^{x+y^2}$

4 equal squares

the sample point to be the upper right-corner

Ans :

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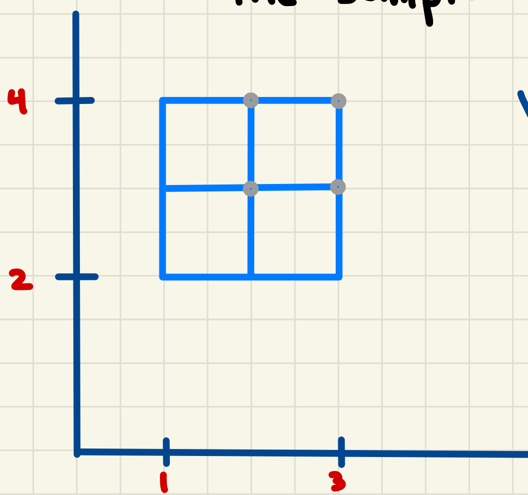

$$V \approx f(2,3)\Delta A + f(2,4)dA + f(3,3)dA + f(3,4)dA$$

$$= e^{11} + e^{18} + e^{12} + e^{19}$$

Example: evaluate $\iint_R f(x,y)\,dA$ exactly if

$R = \{ (x,y) : |x| \leq 2, |y| \leq 3 \}$ and

$f(x,y) = \sqrt{9 - y^2}$

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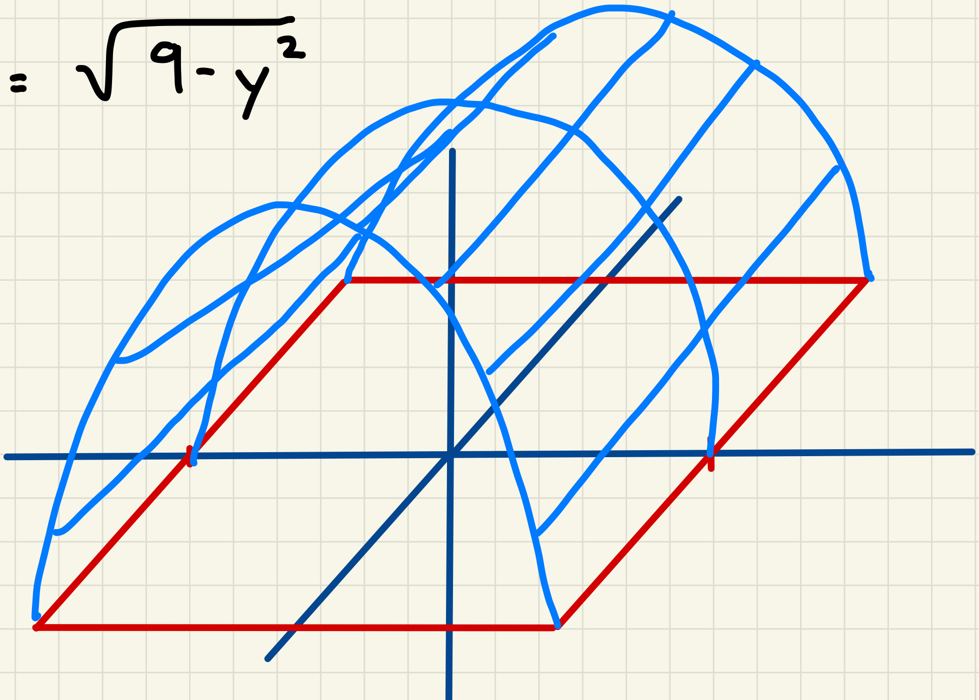


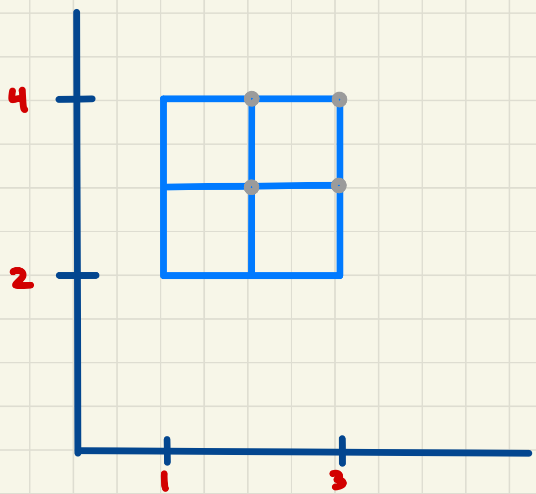
Ans:

$$\frac{\pi(3)^2}{2} \cdot 4$$

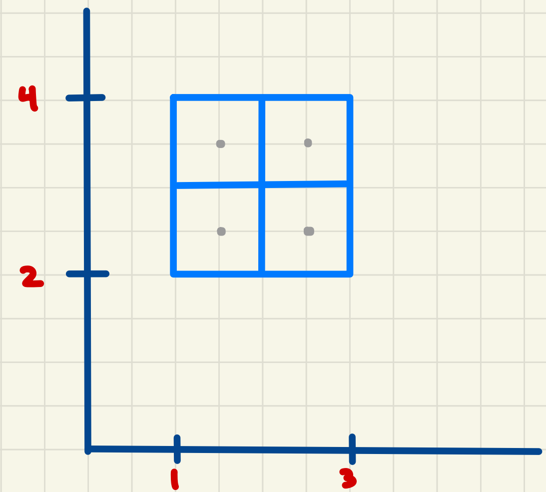
$$= 18\pi$$

Midpoint rule: instead of choosing the upper right-hand corner of the sub-rectangle $R_{i,j}$ to be the sample point, choose the middle/center

Upper right-hand corner



Midpoint



Example: use the midpoint rule to estimate

$$\iint_R x^2 + \sqrt{y} \, dA \quad \text{for} \quad R = \{ (x,y) : \ |x| \le 1 , \ 2 \le y \le 6 \}$$

using 4 rectangles of equal size

Ans:

Example: use the midpoint rule to estimate

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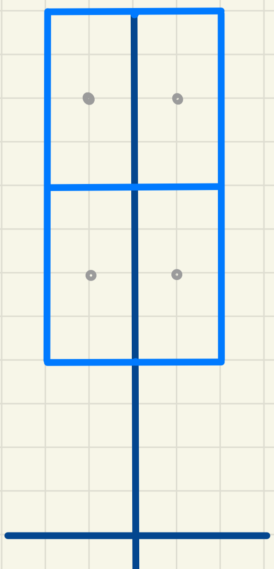
Ans: 

$$V \approx f\left(\tfrac{1}{2},3\right)dA + f\left(\tfrac{1}{2},5\right)dA + f\left(\tfrac{-1}{2},3\right)dA + f\left(\tfrac{-1}{2},5\right)dA$$

$$= \left(\tfrac{1}{4}+\sqrt{3}\right)2 + \left(\tfrac{1}{4}+\sqrt{5}\right)2 + \left(\tfrac{1}{4}+\sqrt{3}\right)2 + \left(\tfrac{1}{4}+\sqrt{5}\right)2$$

Average value: consider a function $f(x)$ of a single variable with

$$A = \int_a^b f(x)\,dx$$

recall that $\dfrac{A}{b-a} = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$ is the AVERAGE VALUE of the function on $[a,b]$

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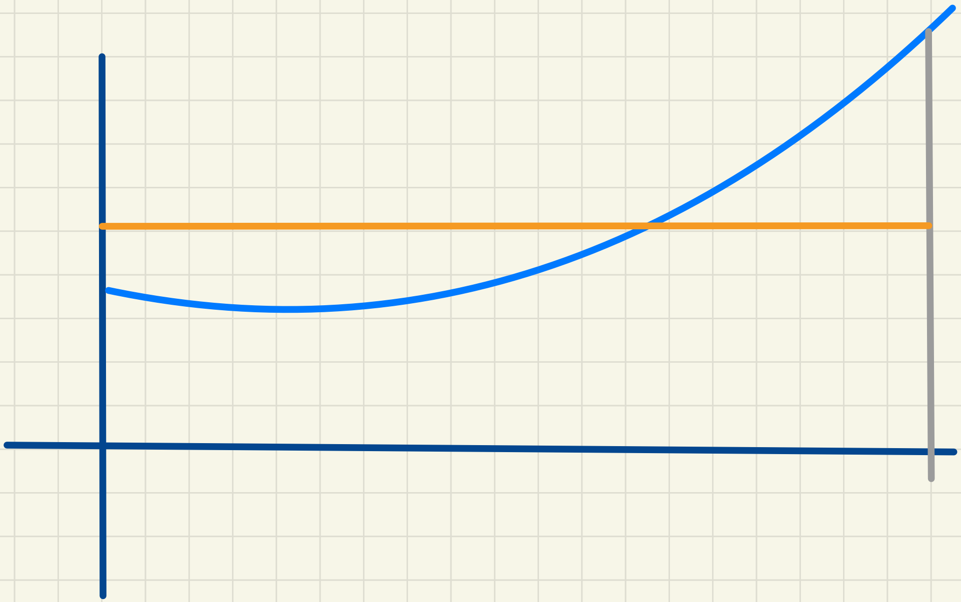

height of the rectangle that would give the same area

Average value: consider a function $f(x,y)$

of two variables on a rectangle R

the AVERAGE VALUE of f on R is

$$\frac{1}{\text{Area}(R)} \iint_R f(x,y)\, dA$$

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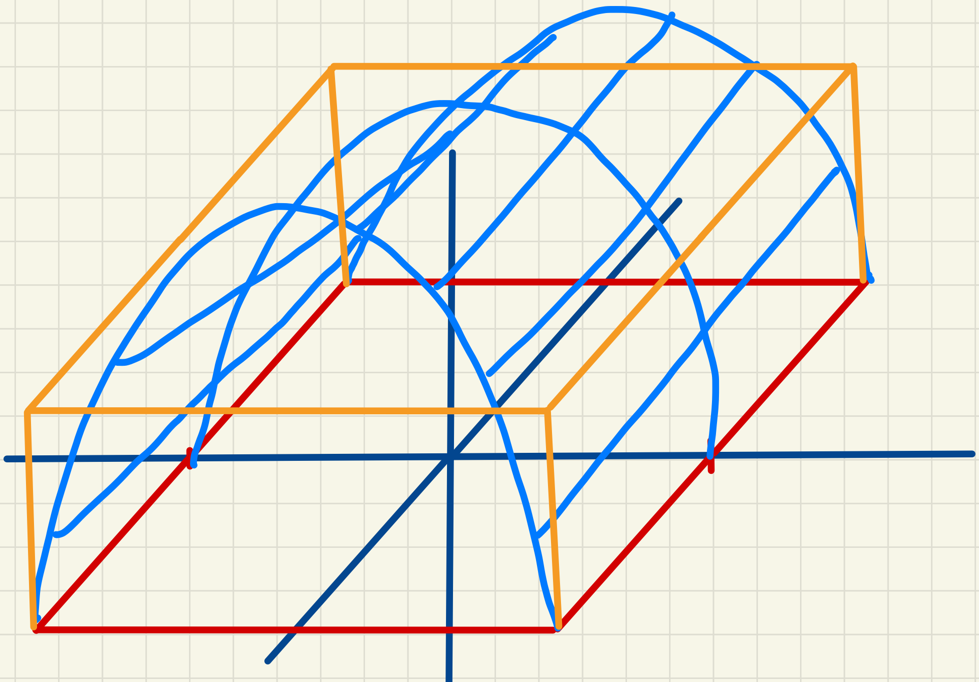

height of the rectangle prism that would give the same volume

Properties of double integrals:

$$\iint_R f(x,y) + g(x,y)\, dA = \iint_R f(x,y)\, dA + \iint_R g(x,y)\, dA$$

$$\iint_R c f(x,y)\, dA = c \iint_R f(x,y)\, dA$$

if $f(x,y) \geq g(x,y)$ on R, then

$$\iint_R f(x,y)\, dA \geq \iint_R g(x,y)\, dA$$