

# Functions of several variables

Previously : vector-valued functions

$$f: \mathbb{R} \rightarrow \mathbb{R}^3$$

$$g: \mathbb{R} \rightarrow \mathbb{R}^2$$

$$h: \mathbb{R} \rightarrow \mathbb{R}^n$$

Now : functions of several variables

$$f: \mathbb{R}^3 \rightarrow \mathbb{R}$$

$$g: \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$h: \mathbb{R}^n \rightarrow \mathbb{R}$$

Example: the Cobb-Douglas production function

Idea: production output is determined by the amount of labor involved and the amount of capital invested

Model:  $P(L, K) = bL^\alpha K^{1-\alpha}$

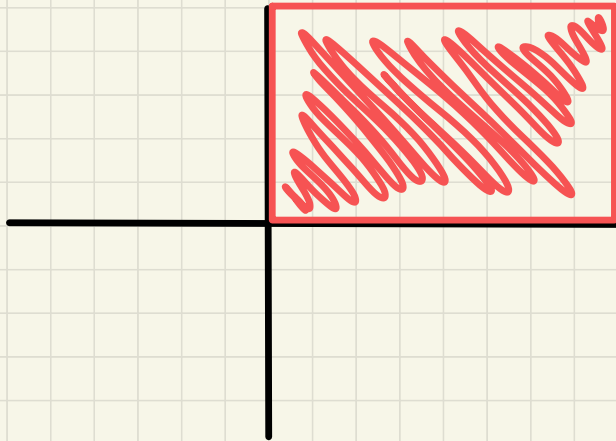
after fitting the historical data:  $P(L, K) = 1.01 L^{.75} K^{.25}$

Example: the Cobb-Douglas production function

Model:  $P(L, K) = bL^\alpha K^{1-\alpha}$

after fitting the historical data:  $P(L, K) = 1.01 L^{.75} K^{.25}$

Domain?  $D = \{ (L, K) : L \geq 0, K \geq 0 \}$



Example: domain and range of

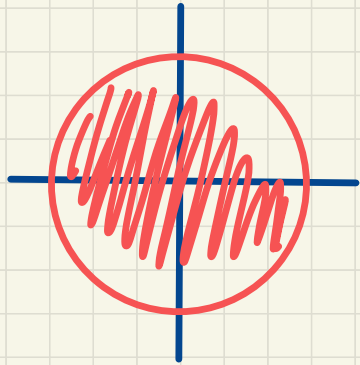
$$g(x,y) = \sqrt{9 - x^2 - y^2}$$

Ans:

Example: domain and range of

$$g(x,y) = \sqrt{9-x^2-y^2}$$

Ans:  $9-x^2-y^2 \geq 0 \iff x^2+y^2 \leq 9$



$$D = \{(x,y) : x^2+y^2 \leq 9\}$$

for the range, note that

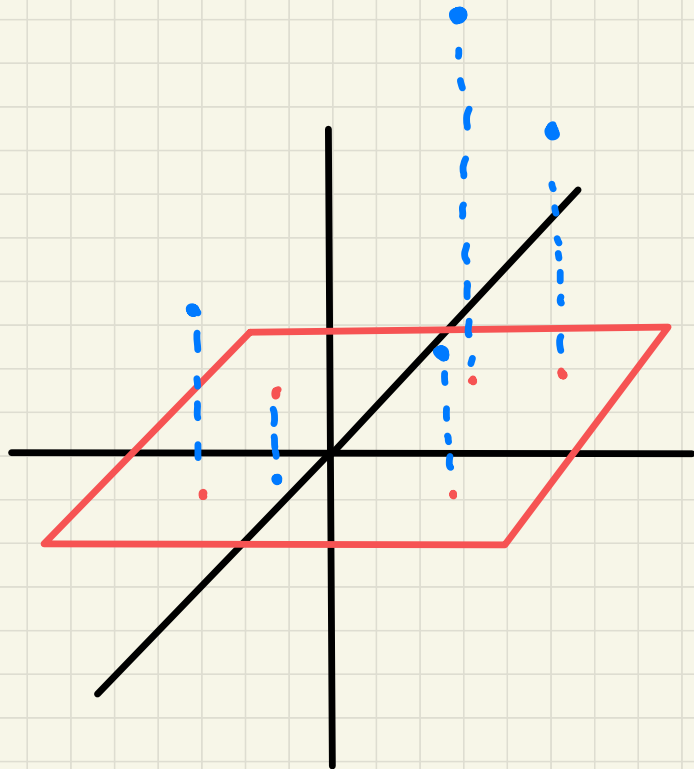
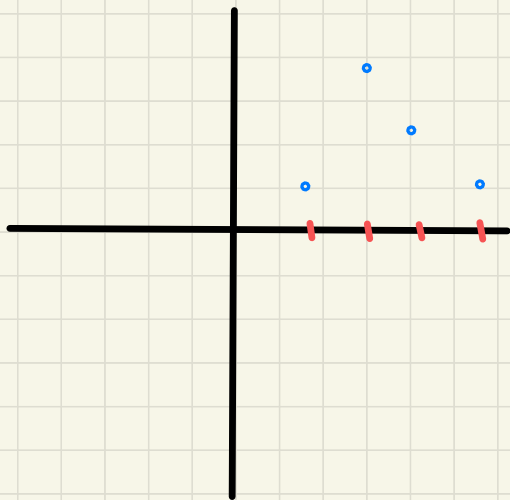
$$0 \leq x^2+y^2 \leq 9 \iff 0 \leq 9-x^2-y^2 \leq 9$$

so  $0 \leq \sqrt{9-x^2-y^2} \leq 3$

$$R = \{z : 0 \leq z \leq 3\}$$

Is there a simple way to understand some crucial aspects of the function?

Graphing!



Example:

$$g(x,y) = \sqrt{9-x^2-y^2}$$

Ans:



Example:

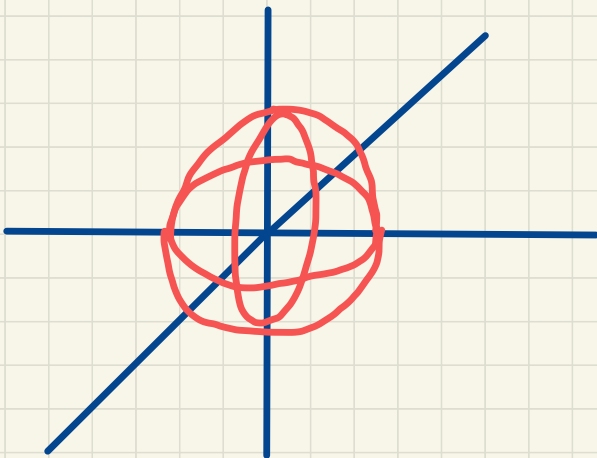
$$g(x,y) = \sqrt{9-x^2-y^2}$$

$$z = \sqrt{9-x^2-y^2}$$

$$z^2 = 9-x^2-y^2$$

$$x^2+y^2+z^2 = 9$$

First (wrong) ans:



why is  
this wrong?

Example:

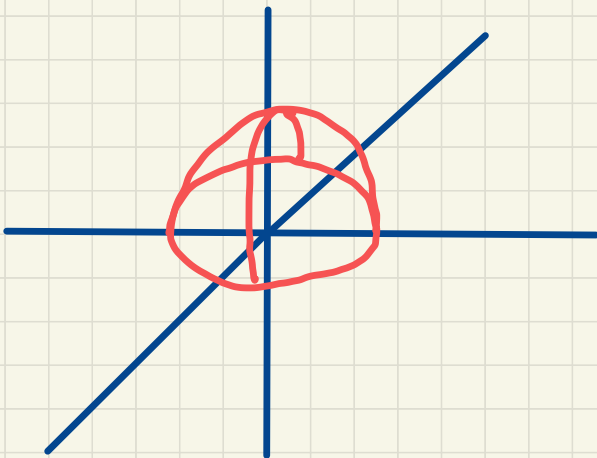
$$g(x,y) = \sqrt{9-x^2-y^2}$$

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$$x^2+y^2+z^2 = 9$$

Ans:



bottom half  
is given by

$$h(x,y) = -\sqrt{9-x^2-y^2}$$

How to draw the graph? Use level curves

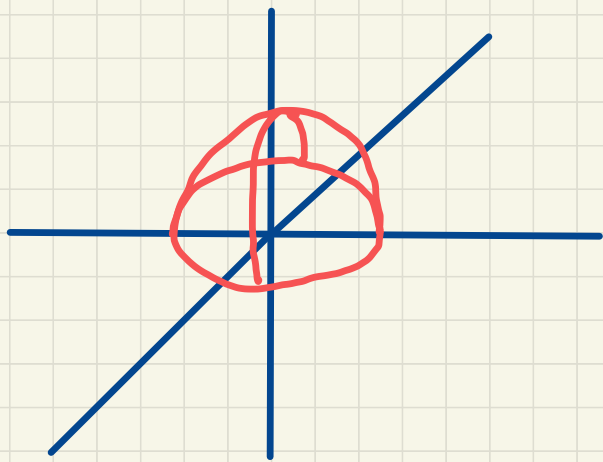
The level curve of a function  $f(x,y)$  is the curve  $f(x,y)=k$  for a fixed constant

Idea: take a slice of a graph  
then piece them together

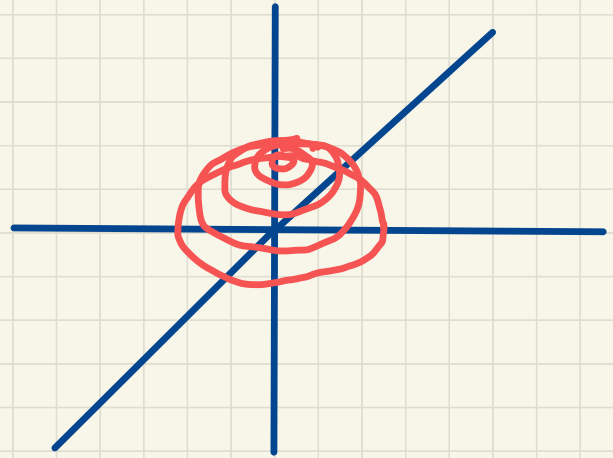
Example:  $f(x,y) = \sqrt{9-x^2-y^2}$

$$0 = \sqrt{9-x^2-y^2} \Leftrightarrow x^2+y^2 = 9$$

$$\sqrt{5} = \sqrt{9-x^2-y^2} \Leftrightarrow x^2+y^2 = 4$$



slice  
it parallel  
to  $xy$ -plane



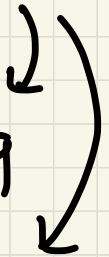
Example :

$$\{(x,y) = \sqrt{9-x^2-y^2}$$

$$0 = \sqrt{9-x^2-y^2} \Leftrightarrow x^2+y^2 = 9$$

$$\sqrt{5} = \sqrt{9-x^2-y^2} \Leftrightarrow x^2+y^2 = 4$$

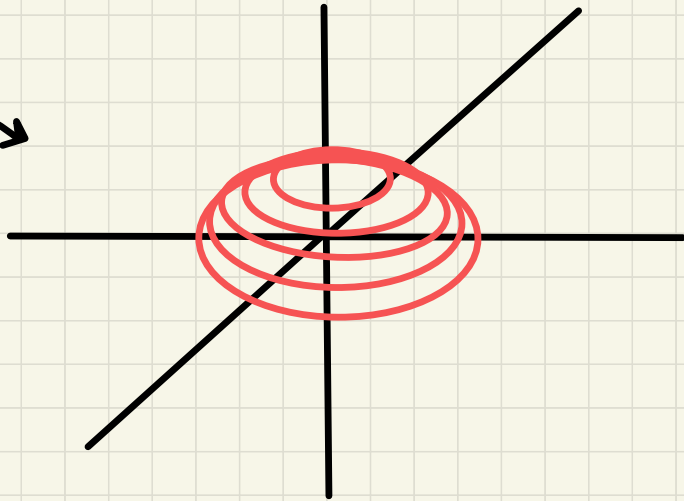
circles



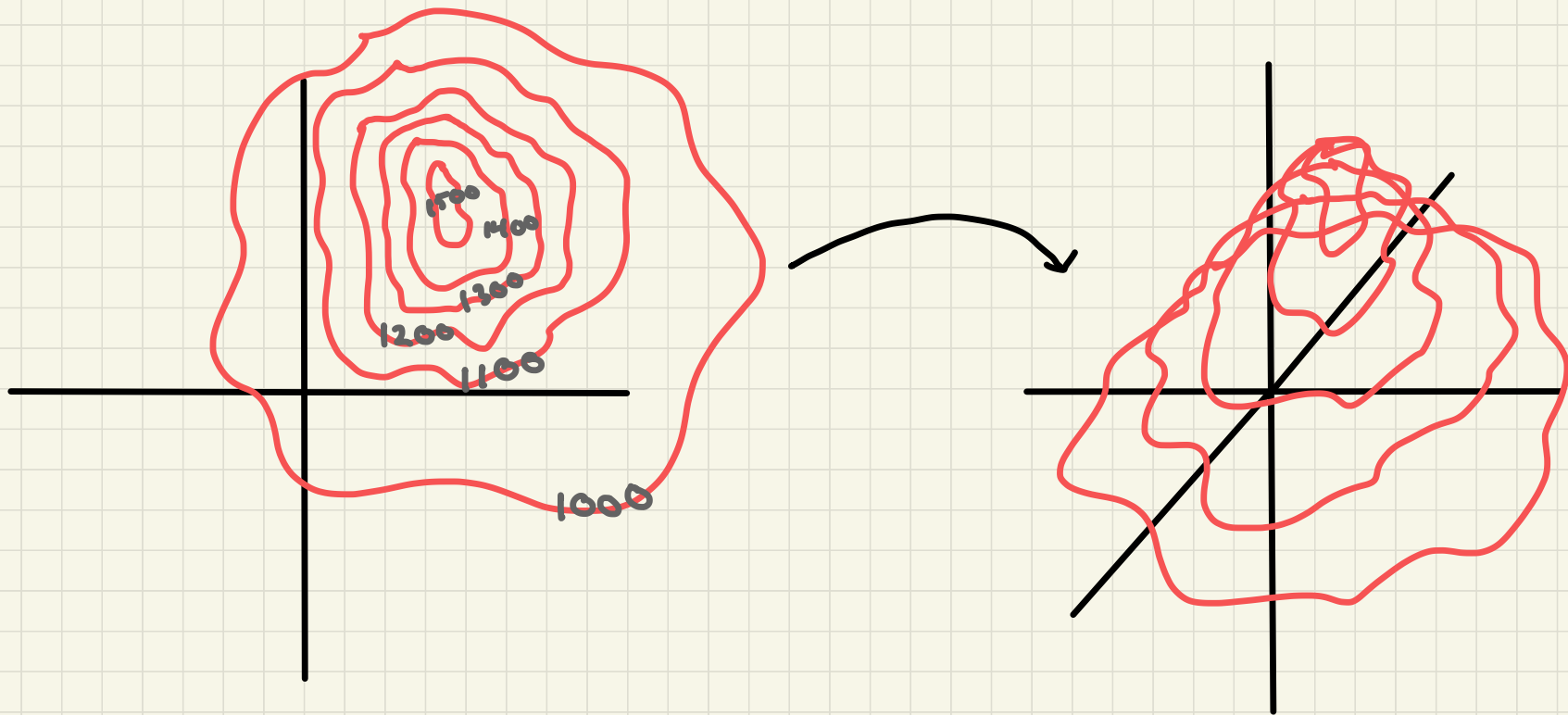
Note : level curves are curves

live in  $\mathbb{R}^2$

$$f(x,y) = \sqrt{9-x^2-y^2}$$



Example: topographic maps



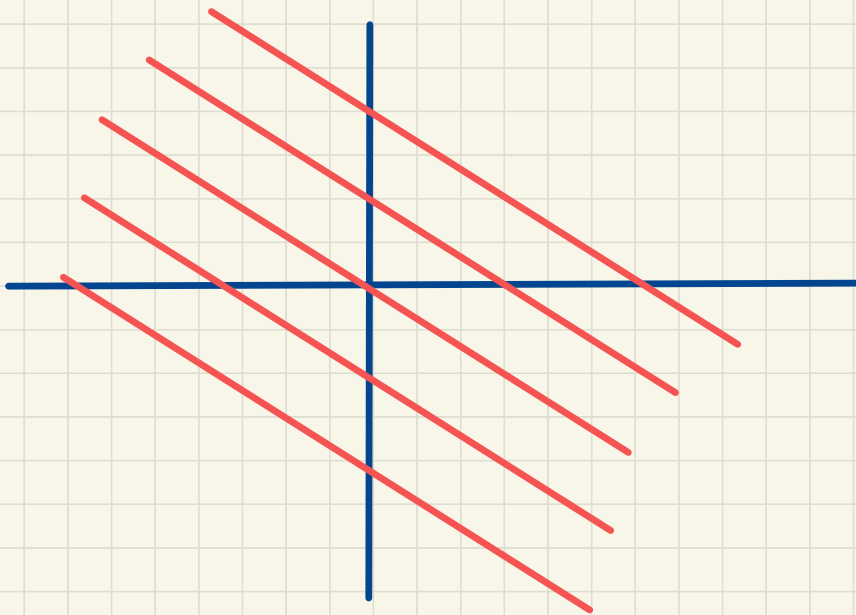
Example: sketch the level curves  
of  $f(x, y) = 1 + 2x + 3y$

Ans:

Example: sketch some level curves

for  $f(x,y) = 1 + 2x + 3y$

Ans:  $k = 1 + 2x + 3y \Leftrightarrow y = \frac{k - 1 - 2x}{3}$





Example: sketch some level curves for

$$f(x,y) = x^2 + 2y^2 + 3$$

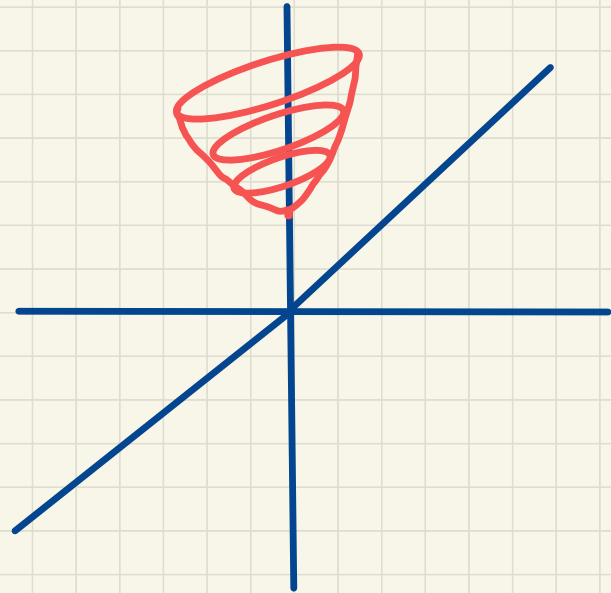
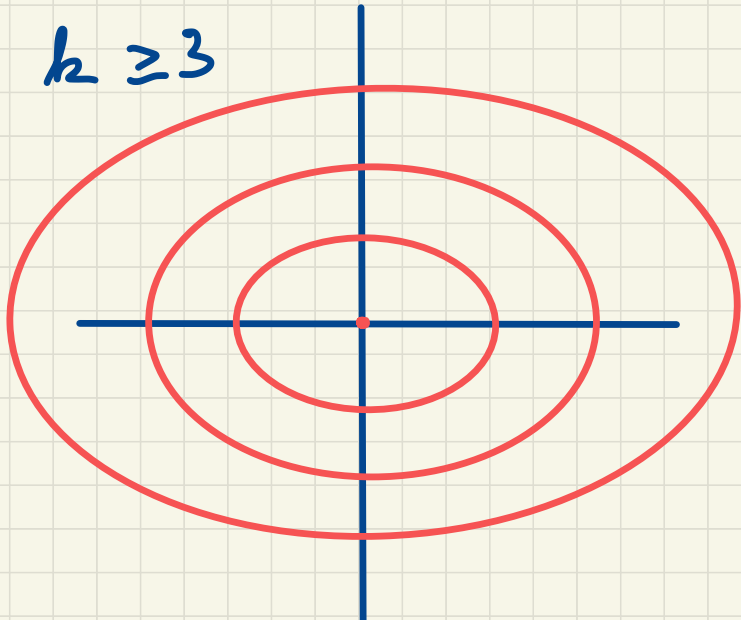
Ans:

Example: sketch some level curves for

$$f(x,y) = x^2 + 2y^2 + 3$$

Ans:  $k = x^2 + 2y^2 + 3 \iff x^2 + 2y^2 = k - 3$

note  $k \geq 3$



## Functions of three or more variables

$$f: \mathbb{R}^n \rightarrow \mathbb{R}$$

more generally,  $f: D \rightarrow \mathbb{R}$

for some  $D \subseteq \mathbb{R}^n$

Example:  $f(x, y, z) = \sqrt{x-y} + \log(z)$

Ans:

## Functions of three or more variables

$$f: \mathbb{R}^n \rightarrow \mathbb{R}$$

more generally,  $f: D \rightarrow \mathbb{R}$

for some  $D \subseteq \mathbb{R}^n$

Example:  $f(x, y, z) = \sqrt{x-y} + \log(z)$

Ans:  $D = \{(x, y, z) : x \geq y \text{ and } z > 0\}$

Problem: graph of  $f: \mathbb{R}^3 \rightarrow \mathbb{R}$

would be four-dimensional

"Solution": still can use the level curve idea

(now level surfaces in  $\mathbb{R}^3$ )

Example: Find the level surfaces of

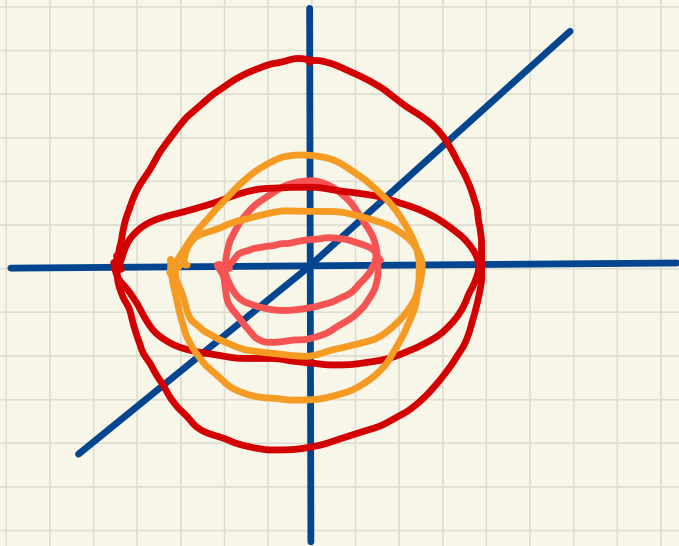
$$f(x, y, z) = x^2 + y^2 + z^2$$

Ans:

Example: Find the level surfaces of

$$f(x, y, z) = x^2 + y^2 + z^2$$

Ans:  $k = x^2 + y^2 + z^2$  is a sphere



Example: a company uses  $n$  different components in making a product and  $c_i$  is the cost per unit of the  $i$ th component and  $x_i$  is the amount of the  $i$ th component that is used

then the total cost is

$$C(x_1, x_2, \dots, x_n) = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

$$= \vec{c} \cdot \vec{x}$$

$$\vec{c} = \langle c_1, \dots, c_n \rangle$$

$$\vec{x} = \langle x_1, \dots, x_n \rangle$$



A function  $f: D \rightarrow \mathbb{R}$  with  $D \subseteq \mathbb{R}^n$  can be thought of as

- a function of  $n$  real variables  $x_1, \dots, x_n$
- a single point  $(x_1, \dots, x_n) \in \mathbb{R}^n$
- a single  $n$ -dimensional vector  $\langle x_1, \dots, x_n \rangle$

We will use each of these later as appropriate