# Functions of several variables

Previously: vector-valued functions

$f : \mathbb{R} \to \mathbb{R}^3$

$g : \mathbb{R} \to \mathbb{R}^2$

$h : \mathbb{R} \to \mathbb{R}^n$

Now: functions of several variables

$f : \mathbb{R}^3 \to \mathbb{R}$

$g : \mathbb{R}^2 \to \mathbb{R}$

$h : \mathbb{R}^n \to \mathbb{R}$

Example: the Cobb-Douglas production Function

Idea: production output is determined by the amount of labor involved and the amount of capital invested

Model: $P(L,K) = bL^{\alpha}K^{1-\alpha}$

after fitting the historical data: $P(L,K) = 1.01L^{.75}K^{.25}$

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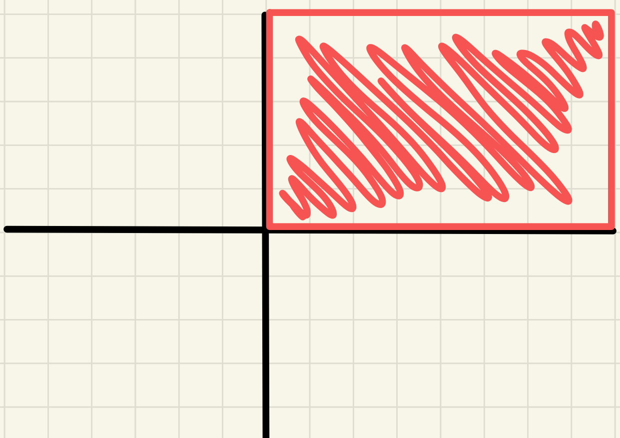
after fitting the historical data: $P(L,K) = 1.01L^{.75}K^{.25}$

Domain? $D = \{(L,K) : L \geq 0,\ K \geq 0\}$



Example: domain and range of

$$g(x,y) = \sqrt{9 - x^2 - y^2}$$

Ans:

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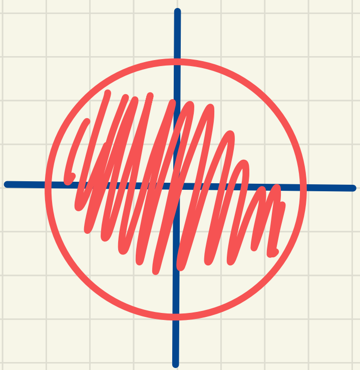
$$g(x,y) = \sqrt{9 - x^2 - y^2}$$

Ans: $9 - x^2 - y^2 \geq 0 \iff x^2 + y^2 \leq 9$



$D = \{(x,y) : x^2 + y^2 \leq 9\}$

for the range, note that

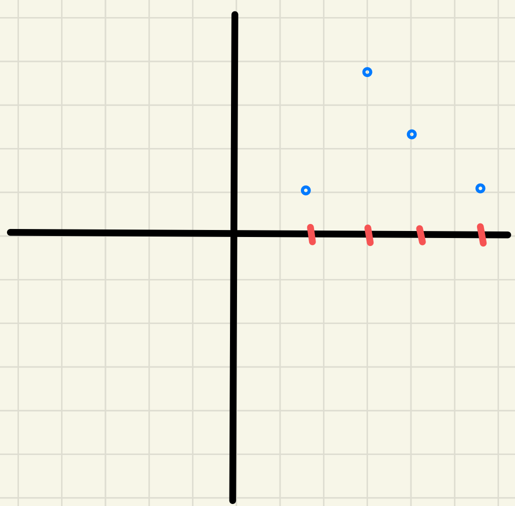
$0 \leq x^2 + y^2 \leq 9 \iff 0 \leq 9 - x^2 - y^2 \leq 9$

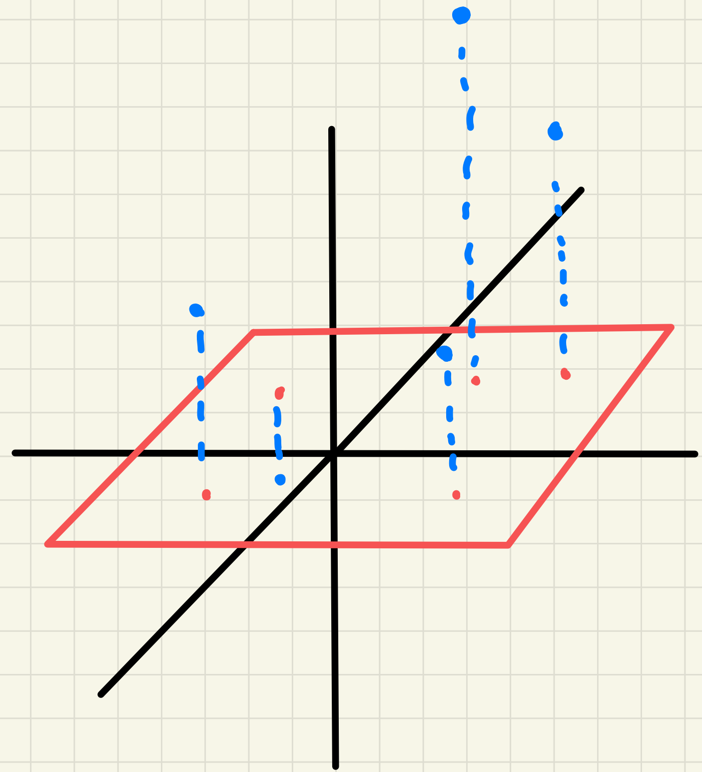
so $0 \leq \sqrt{9 - x^2 - y^2} \leq 3$

$R = \{z : 0 \leq z \leq 3\}$

Is there a simple way to understand some crucial aspects of the function?

Graphing!





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Example:

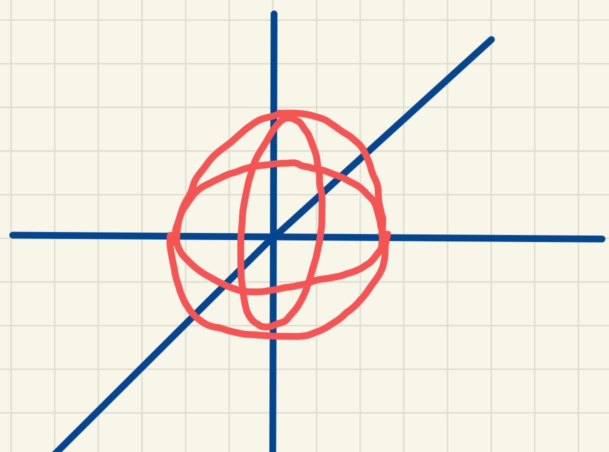
$$g(x,y) = \sqrt{9 - x^2 - y^2}$$

$$z = \sqrt{9 - x^2 - y^2}$$

$$z^2 = 9 - x^2 - y^2$$

$$x^2 + y^2 + z^2 = 9$$

First (wrong) ans:



why is this wrong?

Example:

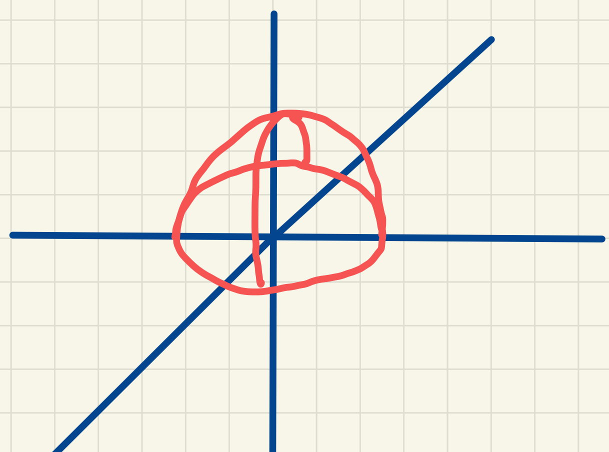
$$g(x,y) = \sqrt{9 - x^2 - y^2}$$

$$z = \sqrt{9 - x^2 - y^2}$$

$$z^2 = 9 - x^2 - y^2$$

$$x^2 + y^2 + z^2 = 9$$

Ans:



bottom half is given by

$$h(x,y) = -\sqrt{9 - x^2 - y^2}$$

How to draw the graph? Use level curves

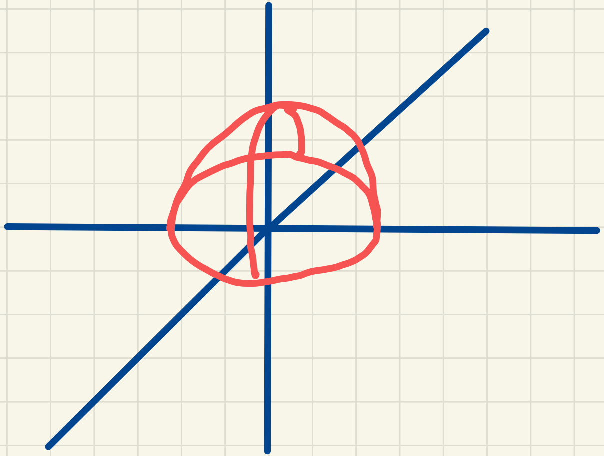
The level curve of a function $f(x,y)$ is the curve $f(x,y)=k$ for a fixed constant

Idea: take a slice of a graph then piece them together

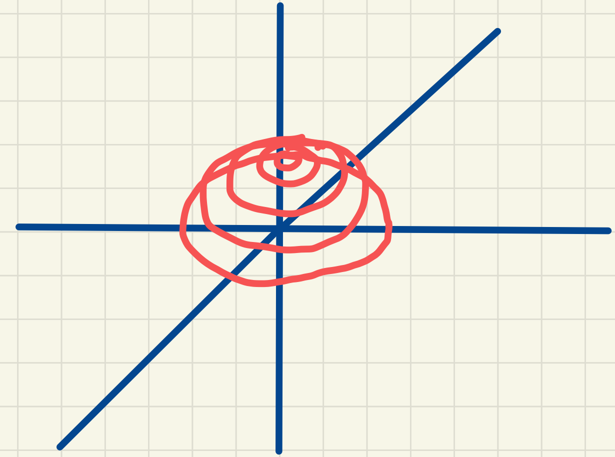
Example: $f(x,y)=\sqrt{9-x^2-y^2}$

$$0=\sqrt{9-x^2-y^2} \iff x^2+y^2=9$$

$$\sqrt{5}=\sqrt{9-x^2-y^2} \iff x^2+y^2=4$$



slice it parallel to xy-plane



circles

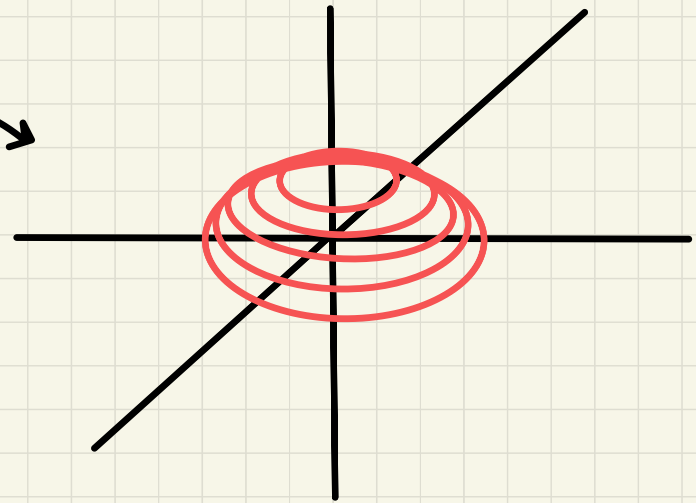
Example: $f(x,y) = \sqrt{9-x^2-y^2}$

$$0 = \sqrt{9-x^2-y^2} \iff x^2+y^2=9$$

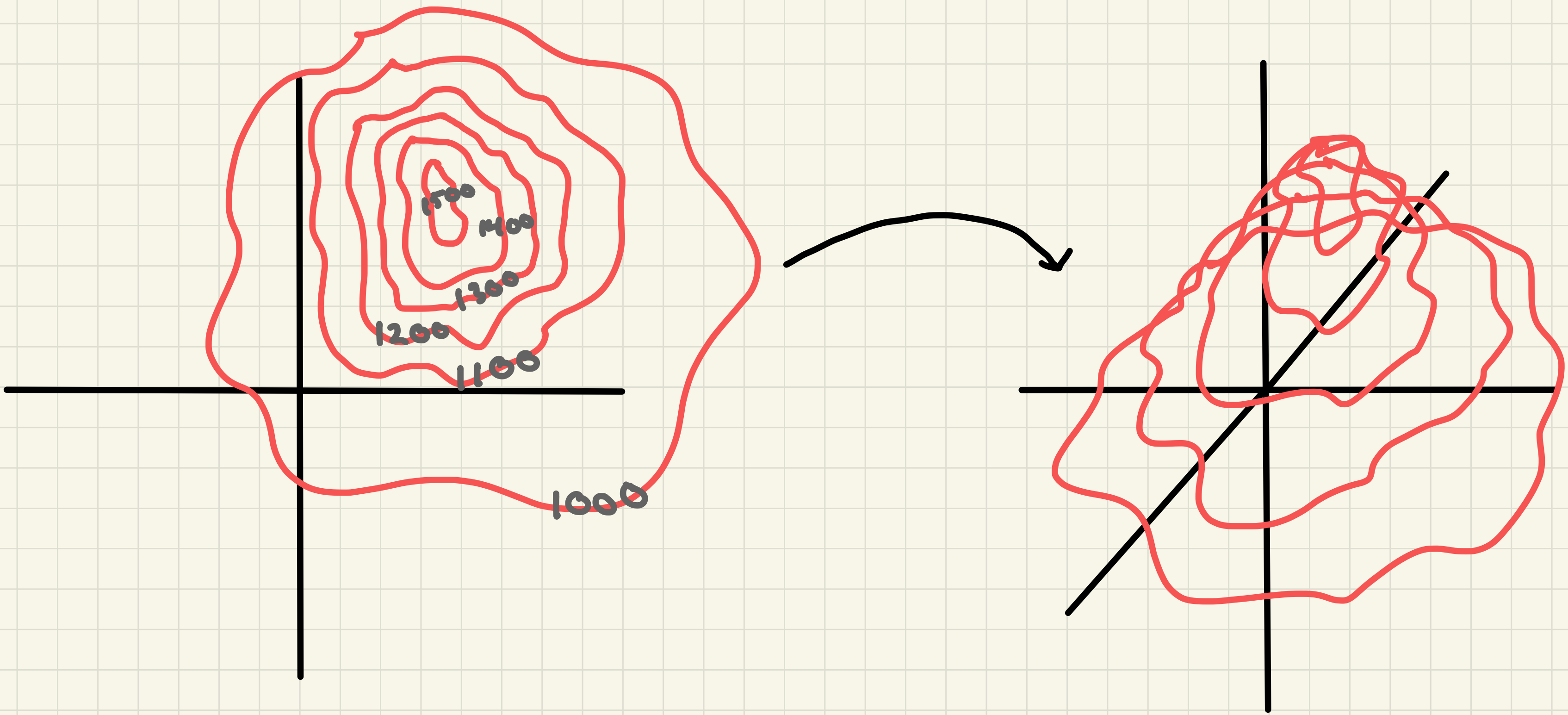
$$\sqrt{5} = \sqrt{9-x^2-y^2} \iff x^2+y^2=4$$

Note: level curves are curves live in $\mathbb{R}^2$

$$f(x,y) = \sqrt{9 - x^2 - y^2}$$

# Example: topographic maps

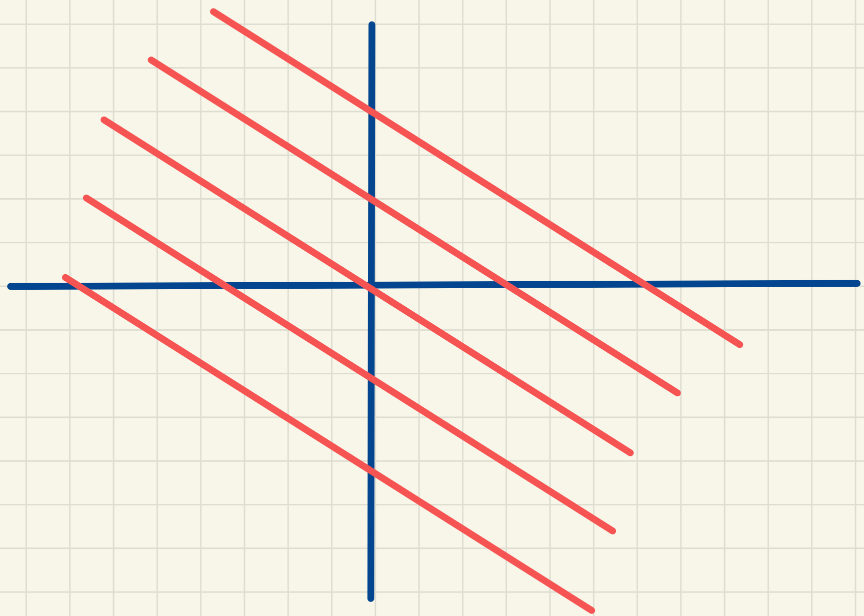


Example: sketch the level curves of $f(x,y) = 1+2x+3y$

Ans:

Example: sketch some level curves for $f(x,y) = 1+2x+3y$

Ans: $k = 1+2x+3y \iff y = \dfrac{k-1-2x}{3}$



Example: sketch some level curves for

$$f(x,y) = 5x^2 + 2y^2 + 3$$

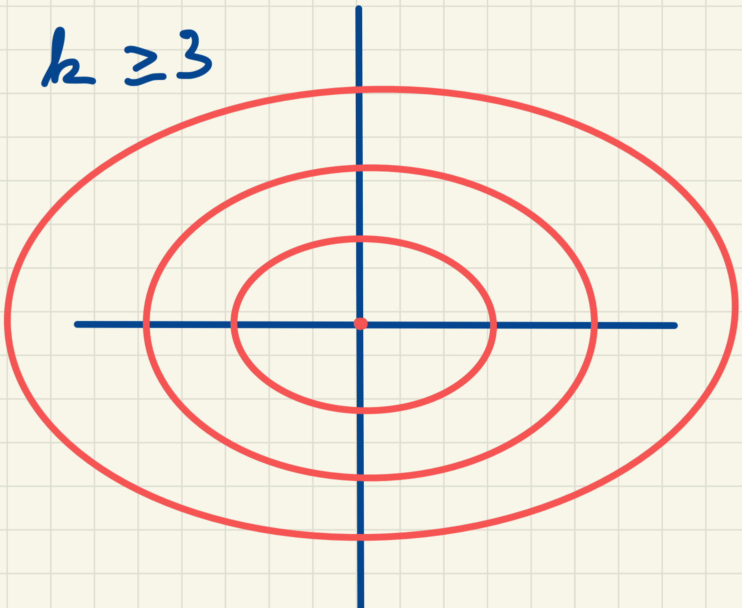
Ans:

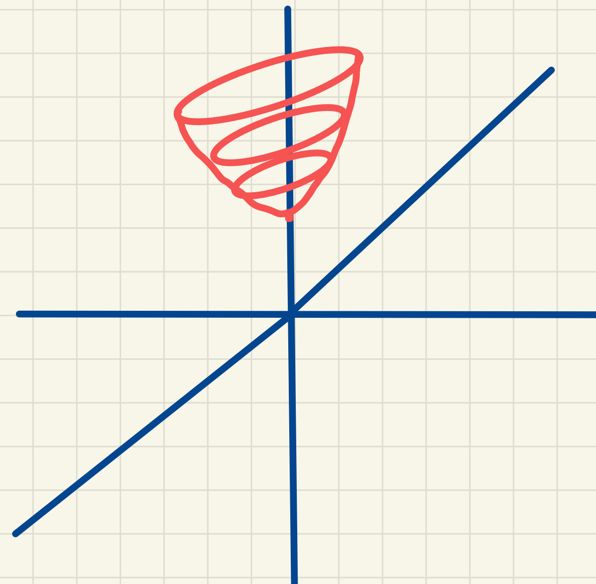
Example: sketch some level curves for

$f(x,y) = x^2 + 2y^2 + 3$

Ans: $k = x^2 + 2y^2 + 3 \iff x^2 + 2y^2 = k - 3$

note $k \geq 3$





# Functions of three or more variables

$f : \mathbb{R}^n \to \mathbb{R}$

more generally, $f : D \to \mathbb{R}$

for some $D \subseteq \mathbb{R}^n$

Example: $f(x,y,z) = \sqrt{x-y} + \log(z)$

Ans :

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$f: \mathbb{R}^n \to \mathbb{R}$

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for some $D \subseteq \mathbb{R}^n$

Example: $f(x,y,z) = \sqrt{x-y} + \log(z)$

Ans: $D = \{(x,y,z) : x \geq y \text{ and } z > 0\}$

Problem: graph of $f: \mathbb{R}^3 \to \mathbb{R}$

would be four-dimensional

"Solution": still can use the level curve idea

(now level surfaces in $\mathbb{R}^3$)

Example: Find the level surfaces of

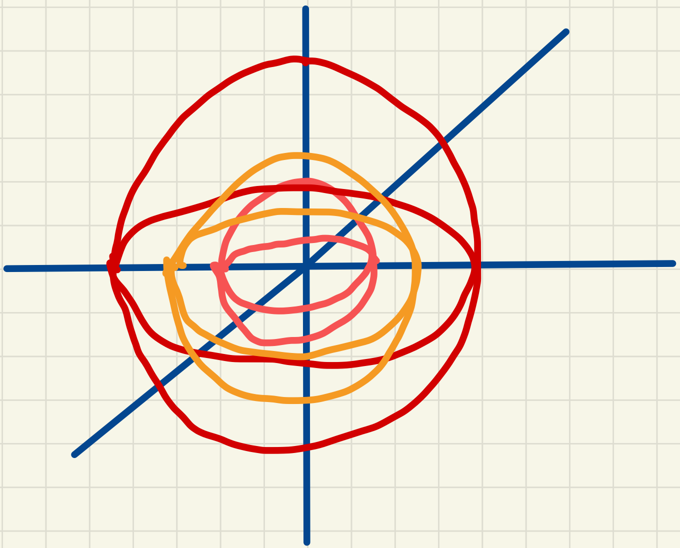
$$f(x, y, z) = x^2 + y^2 + z^2$$

Ans:

Example: Find the level surfaces of

$$f(x,y,z) = x^2+y^2+z^2$$

Ans: $k = x^2+y^2+z^2$ is a sphere



Example: a company uses $n$ different components in making a product and $c_i$ is the cost per unit of the ith component and $x_i$ is the amount of the ith component that is used

then the total cost is

$$C(x_1, x_2, \dots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$
$$= \vec{c} \cdot \vec{x}_n$$

A function $f: D \to \mathbb{R}$ with $D \subseteq \mathbb{R}^n$ can be thought of as

- a function of $n$ real variables $x_1, \ldots, x_n$
- a single point $(x_1, \ldots, x_n) \in \mathbb{R}^n$
- a single $n$-dimensional vector $\langle x_1, \ldots, x_n \rangle$

We will use each of these later as appropriate