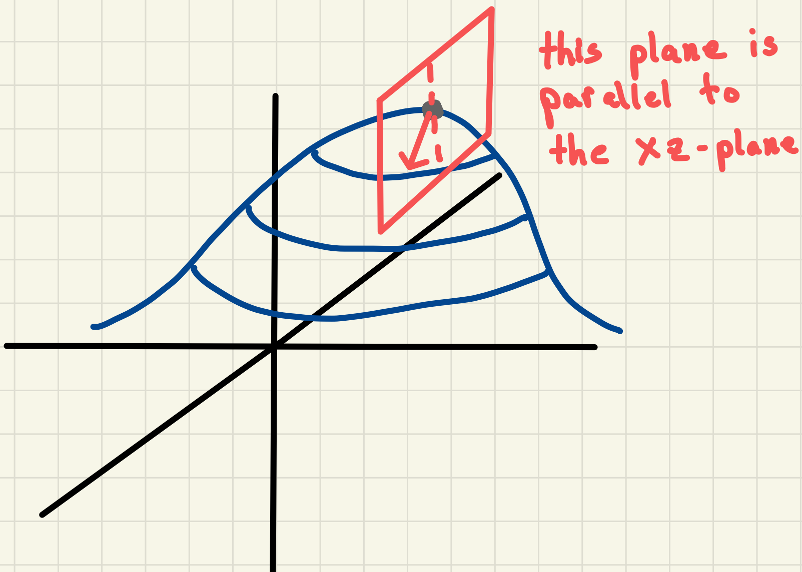
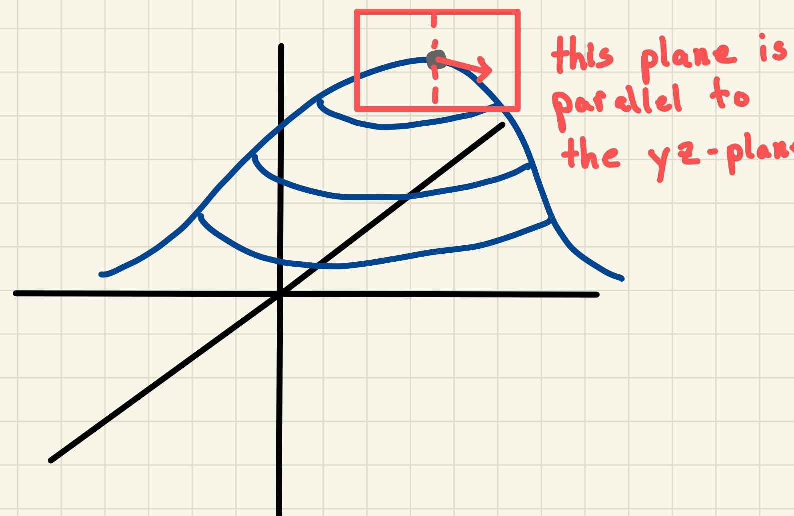
Directional derivatives and the gradient vector

Recall the interpretation of partial derivatives:

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

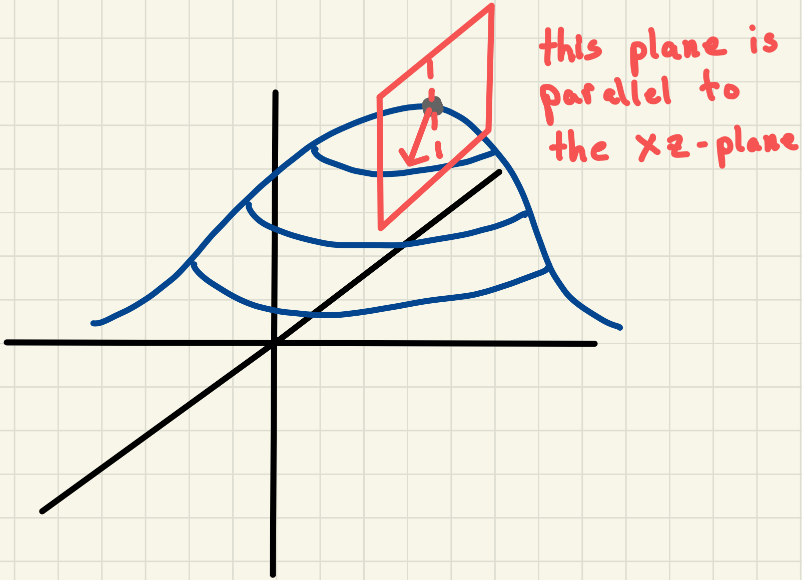


$$f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$$



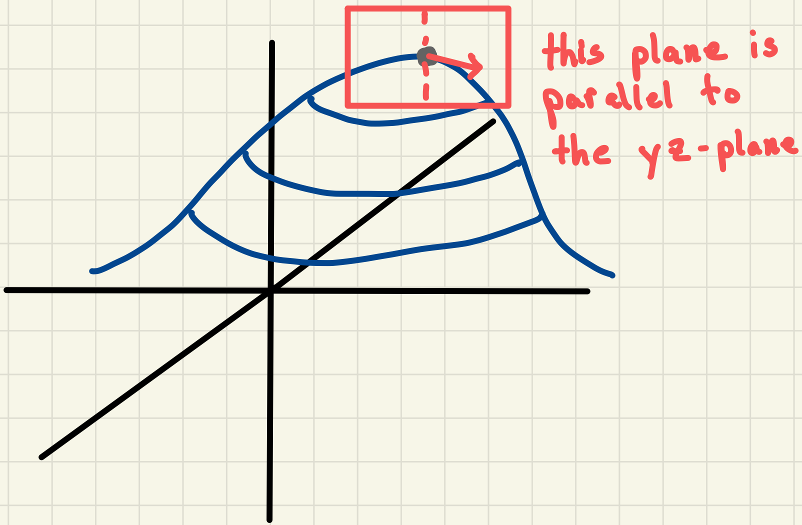
Directional derivative in the direction of $\langle 1,0 \rangle$

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

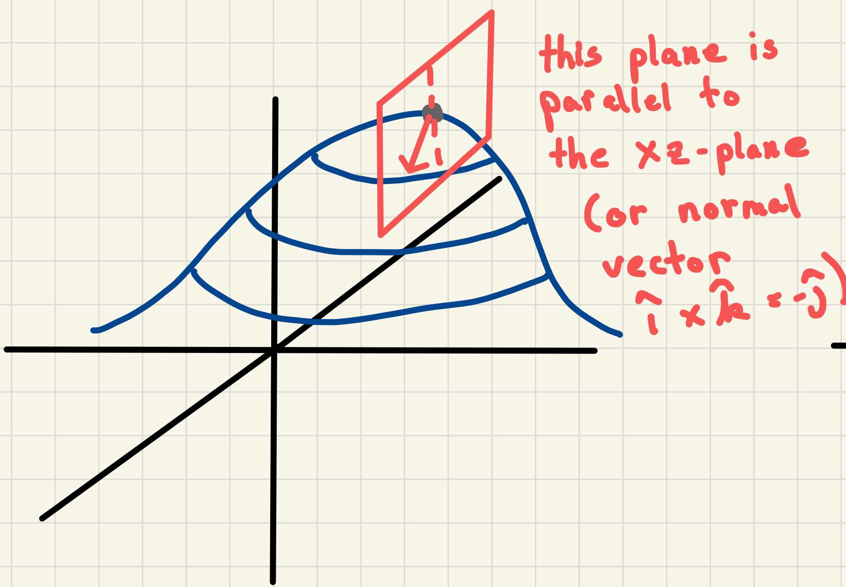


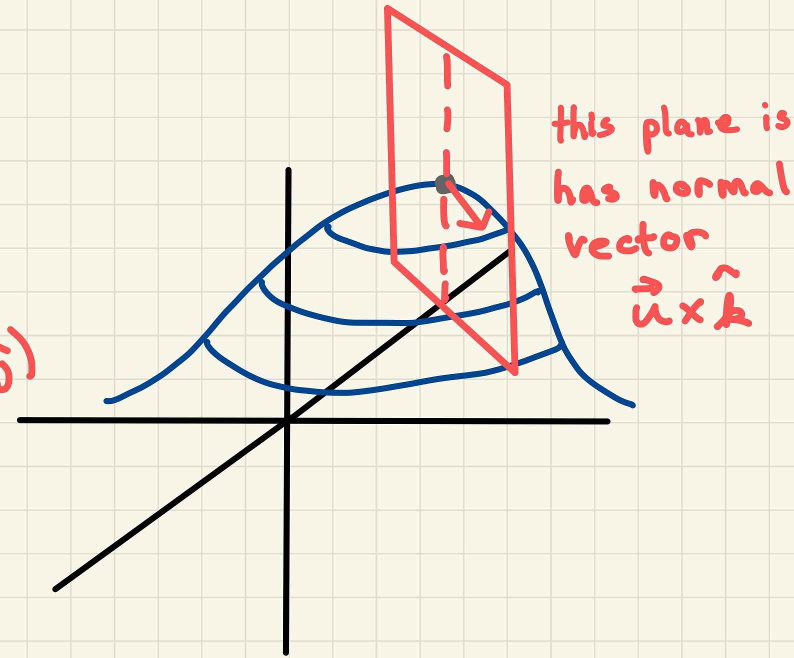
Directional derivative in the direction of $\langle 0,1 \rangle$

$$f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$$



Natural question: what about the directional derivative in the direction of a general unit vector $\vec{u} \in \mathbb{R}^2$





if $\vec{u} = \langle a, b \rangle$ is a unit vector, then

the directional derivative of f at (x_0, y_0)

in the direction of $\vec{u}$ is

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}$$

Example: if $\vec{u} = \langle 1, 0 \rangle$, $D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0)$

if $\vec{u} = \langle 0, 1 \rangle$, $D_{\vec{u}} f(x_0, y_0) = f_y(x_0, y_0)$

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}$$

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Theorem: if $f(x,y)$ is a differentiable function of x and y and $\vec{u} = \langle a, b \rangle$ is a unit vector,

$$D_{\vec{u}} f(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle \cdot \langle a, b \rangle$$

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}$$

Example: if $\vec{u} = \langle 1, 0 \rangle$, $D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0)$

if $\vec{u} = \langle 0, 1 \rangle$, $D_{\vec{u}} f(x_0, y_0) = f_y(x_0, y_0)$

Theorem: if $f(x, y)$ is a differentiable function of x and y and $\vec{u} = \langle a, b \rangle$ is a unit vector,

$$D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0)\, a + f_y(x_0, y_0)\, b$$

Theorem: if $f(x,y)$ is a differentiable function of x and y and $\vec{u} = \langle a,b \rangle$ is a unit vector,

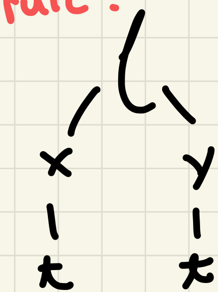
$$D_{\vec{u}} f(x_0,y_0) = \langle f_x(x_0,y_0), f_y(x_0,y_0) \rangle \cdot \langle a, b \rangle$$

Proof: $$D_{\vec{u}} f(x_0,y_0) = \lim_{h \to 0} \frac{f(x_0+ah, y_0+bh) - f(x_0,y_0)}{h}$$

let $g(t) = f(x_0+at, y_0+bt)$

$$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = D_{\vec{u}} f(x_0,y_0)$$

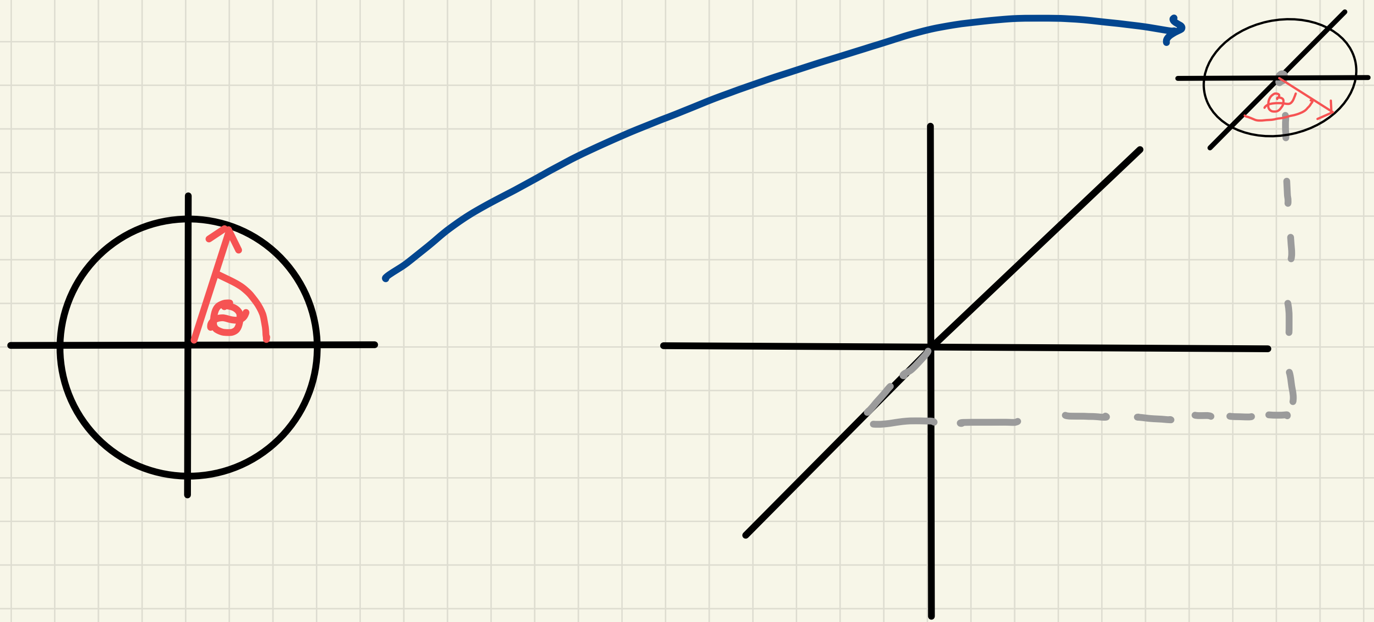
chain rule!



$$g'(0) = f_x(x_0,y_0)a + f_y(x_0,y_0)b = \langle f_x(x_0,y_0), f_y(x_0,y_0) \rangle \cdot \langle a,b \rangle$$

Any unit vector $\langle a, b\rangle$ can be written

as $\langle a, b\rangle = \langle \cos(\theta), \sin(\theta)\rangle$ for some $\theta \in [0, 2\pi)$



$$D_{\vec{u}} f(x, y) = f_x(x, y)\cos(\theta) + f_y(x, y)\sin(\theta)$$

Example: find $D_{\vec{u}} f(x,y)$ if $f(x,y) = e^x y + \ln(x+y)$

and $\vec{u}$ is the unit vector given by angle $\theta = \frac{\pi}{4}$

what is $D_{\vec{u}} f(3,1)$?

Ans:

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what is $D_{\vec{u}} f(3,1)$?

Ans: $f_x(x,y) = e^x y + \frac{1}{x+y}$ $\qquad$ $f_y(x,y) = e^x + \frac{1}{x+y}$

$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ $\qquad$ $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$$D_{\vec{u}} f(x,y) = \left(e^x y + \frac{1}{x+y}\right)\frac{\sqrt{2}}{2} + \left(e^x + \frac{1}{x+y}\right)\frac{\sqrt{2}}{2}$$

$$D_{\vec{u}} f(3,1) = \left(e^3 + \frac{1}{4}\right)\frac{\sqrt{2}}{2} + \left(e^3 + \frac{1}{4}\right)\frac{\sqrt{2}}{2}$$

$$D_{\vec{u}}f(x,y) = \underbrace{\langle f_x(x,y), f_y(x,y)\rangle}_{\text{the gradient } \nabla f} \cdot \langle a,b\rangle$$

in other words,

$$D_{\vec{u}}f(x,y) = \nabla f \cdot \vec{u}$$

is a vector!

the vector of partial derivatives

Example: if $f(x,y) = e^{x^2+xy^2+3y^4} + \cos(y)$,

find $\nabla f(x,y)$, $\nabla f(1,0)$, and $D_{\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle} f(1,0)$

Ans:

Example: if $f(x,y) = e^{x^2+xy^2+3y^4} + \cos(y)$,

find $\nabla f(x,y)$, $\nabla f(1,0)$, and $D_{\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle} f(1,0)$

Ans: $\nabla f(x,y) = \langle f_x, f_y \rangle$

$= \langle e^{x^2+xy^2+3y^4}(2x+y^2), e^{x^2+xy^2+3y^4}(2xy+12y^3) - \sin(y) \rangle$

$\nabla f(1,0) = \langle 2e, 0 \rangle$

$D_{\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle} f(1,0) = \nabla f(1,0) \cdot \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle = e$

Example: find the directional derivative of $f(x,y) = x^3y^2 - \ln(x+y^2)$ at $(2,1)$ in the direction of the vector $\vec{v} = 2\vec{i} + 4\vec{j}$

Ans:

Example: find the directional derivative of

$f(x,y) = x^3y^2 - \ln(x+y^2)$ at $(2,1)$ in

the direction of the vector $\vec{v} = 2\vec{i} + 4\vec{j}$

Ans: $\nabla f = \langle f_x, f_y \rangle = \left\langle 3x^2y^2 - \frac{1}{x+y^2},\ 2x^3y - \frac{1}{x+y^2}(2y) \right\rangle$

$$\nabla f(2,1) = \left\langle 12 - \frac{1}{3},\ 16 - \frac{1}{3}2 \right\rangle = \left\langle \frac{35}{3}, \frac{46}{3} \right\rangle$$

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{2\vec{i}+4\vec{j}}{\sqrt{4+16}} = \frac{2\vec{i}+4\vec{j}}{2\sqrt{5}} = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$$

$$D_{\vec{u}} f(2,1) = \nabla f(2,1) \cdot \vec{u} = \frac{35}{3}\frac{1}{\sqrt{5}} + \frac{46}{3}\frac{2}{\sqrt{5}}$$

Functions of three variables

if $f(x,y,z)$, we can again define the directional derivative $D_{\vec{u}}f(x,y,z)$ for $\vec{u}$ a unit vector in $\mathbb{R}^3$

if $\vec{u} = \langle a,b,c \rangle$,

$$D_{\vec{u}}f(x_0,y_0,z_0) = \lim_{h \to 0} \frac{f(x_0+ah, y_0+bh, z_0+ch) - f(x_0,y_0,z_0)}{h}$$

if $\vec{u} = \langle a, b, c \rangle$,

$$D_{\vec{u}} f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh, z_0 + ch) - f(x_0, y_0, z_0)}{h}$$

if $\vec{v}_0 = \langle x_0, y_0, z_0 \rangle$, we can write this as

$$D_{\vec{u}} f(\vec{v}_0) = \lim_{h \to 0} \frac{f(\vec{v}_0 + h\vec{u}) - f(\vec{v}_0)}{h}$$

if $\vec{u} = \langle a, b, c \rangle$,

$$D_{\vec{u}} f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh, z_0 + ch) - f(x_0, y_0, z_0)}{h}$$

$\nabla f(x, y, z) = \langle f_x, f_y, f_z \rangle$ is the gradient

Theorem: $D_{\vec{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \vec{u}$

Proof: chain rule again!

Example: if $f(x,y,z) = xe^{yz^2}$

find $\nabla f(x,y,z)$ and the directional derivative

at $(2,1,2)$ in the direction of $\vec{v} = \langle 3, 4, 5 \rangle$

Ans:

Example: if $f(x,y,z) = xe^{yz^2}$

find $\nabla f(x,y,z)$ and the directional derivative

at $(2,1,2)$ in the direction of $\vec{v} = \langle 3, 4, 5 \rangle$

Ans: $\nabla f = \langle f_x, f_y, f_z \rangle = \langle e^{yz^2}, xe^{yz^2} z^2, xe^{yz^2}(2yz) \rangle$

$\nabla f(2,1,2) = \langle e^4, 8e^4, 8e^4 \rangle$, $\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 3,4,5 \rangle}{\sqrt{9+16+25}} = \left\langle \frac{3}{\sqrt{50}}, \frac{4}{\sqrt{50}}, \frac{5}{\sqrt{50}} \right\rangle$

$D_{\vec{u}} f(2,1,2) = \langle e^4, 8e^4, 8e^4 \rangle \cdot \left\langle \frac{3}{\sqrt{50}}, \frac{4}{\sqrt{50}}, \frac{5}{\sqrt{50}} \right\rangle = \frac{75e^4}{\sqrt{50}}$

Maximizing the directional derivative

if $f(x,y)$ (or $f(x,y,z)$), which direction

$\vec{u} \in \mathbb{R}^2$ (or $\vec{u} \in \mathbb{R}^3$) maximizes

$D_{\vec{u}} f(x,y)$ (or $D_{\vec{u}} f(x,y,z)$) ?

what is the value of this maximum?

Theorem: the maximum value of $D_{\vec{u}} f(\vec{x})$ is $|\nabla f(\vec{x})|$ and it occurs when $\vec{u}$ has the same direction as $\nabla f(\vec{x})$

similarly, the minimum value of $D_{\vec{u}} f(\vec{x})$ is $-|\nabla f(\vec{x})|$ and it occurs when $\vec{u}$ has the opposite direction as $\nabla f(\vec{x})$

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Proof:

$$D_{\vec{u}}f(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{u}$$

$$= |\nabla f(\vec{x})| \underbrace{|\vec{u}|}_{=1} \cos(\theta)$$

the angle $\theta \in [0, \pi]$ between $\nabla f(\vec{x})$ and $\vec{u}$

maximum: $\theta = 0$, $\cos(\theta) = 1$, $D_{\vec{u}}f(\vec{x}) = |\nabla f(\vec{x})|$

minimum: $\theta = \pi$, $\cos(\theta) = -1$, $D_{\vec{u}}f(\vec{x}) = -|\nabla f(\vec{x})|$

Interpretation: gradient tells you where to go to increase your altitude as quickly as possible

going in the opposite direction decreases your altitude as quickly as possible

Example: if $f(x,y) = xy^2 + \frac{1}{y}$

find the rate of change of f at $\overset{=P}{(2,1)}$ in the direction from $(2,1)$ to $(1,2)$

in what direction does f have the maximum rate of change at P? what is this rate?

Ans:

Example: if $f(x,y) = xy^2 + \frac{1}{y}$ find the rate of change of f at $P = (2,1)$ in the direction from $P = (2,1)$ to $Q = (1,2)$

in what direction does f have the maximum rate of change at P? what is this rate?

Ans: $\nabla f = \langle f_x, f_y \rangle = \langle y^2, 2xy - \frac{1}{y^2} \rangle$

$\nabla f(2,1) = \langle 1,3 \rangle$, $\quad \vec{PQ} = \langle -1,1 \rangle$, $\quad \vec{u} = \frac{\vec{PQ}}{|\vec{PQ}|} = \langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

Example: if $f(x,y) = xy^2 + \frac{1}{y}$ find the rate of change of f at $P = (2,1)$ in the direction from $P = (2,1)$ to $Q = (1,2)$

in what direction does f have the maximum rate of change at P? what is this rate?

Ans: $\nabla f(2,1) = \langle 1, 3 \rangle$ $\vec{u} = \frac{\vec{PQ}}{|\vec{PQ}|} = \langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$D_{\vec{u}} f(2,1) = \nabla f(2,1) \cdot \vec{u} = \frac{2}{\sqrt{2}} = \sqrt{2}$

Example: if $f(x,y) = xy^2 + \frac{1}{y}$

in what direction does f have the maximum rate of change? what is this rate?

Ans: $\nabla f(2,1) = \langle 1, 3 \rangle$

the direction of maximal rate of change is

$\frac{\nabla f(2,1)}{|\nabla f(2,1)|} = \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle$ and this rate is

$|\nabla f(2,1)| = \sqrt{10}$

Example: suppose the temperature in a room is given by $T(x,y,z) = \dfrac{75}{1+2x^2+3y^2+4\sqrt{z}}$

in which direction is the temperature decreasing the fastest at the point $(1,-1,1)$? what is this rate?

Ans:

Example: suppose the temperature in a room is given by $T(x,y,z) = \dfrac{75}{1+2x^2+3y^2+4\sqrt{z}}$

in which direction is the temperature decreasing the fastest at the point $(1,-1,1)$? what is this rate?

Ans: $\nabla T = \langle T_x, T_y, T_z \rangle$

$$= \left\langle \frac{-75(4x)}{(1+2x^2+3y^2+4\sqrt{z})^2}, \frac{-75(6y)}{(1+2x^2+3y^2+4\sqrt{z})^2}, \frac{-75\left(\frac{4}{2\sqrt{z}}\right)}{(1+2x^2+3y^2+4\sqrt{z})^2} \right\rangle$$

$$\nabla T(1,-1,1) = \left\langle -3, \frac{9}{2}, \frac{-3}{2} \right\rangle$$

Example: suppose the temperature in a room is given by $T(x,y,z) = \dfrac{75}{1+2x^2+3y^2+4\sqrt{z}}$

in which direction is the temperature decreasing the fastest at the point $(1,-1,1)$? what is this rate?

Ans: $\nabla T(1,-1,1) = \langle -3, \frac{9}{2}, -\frac{3}{2} \rangle$

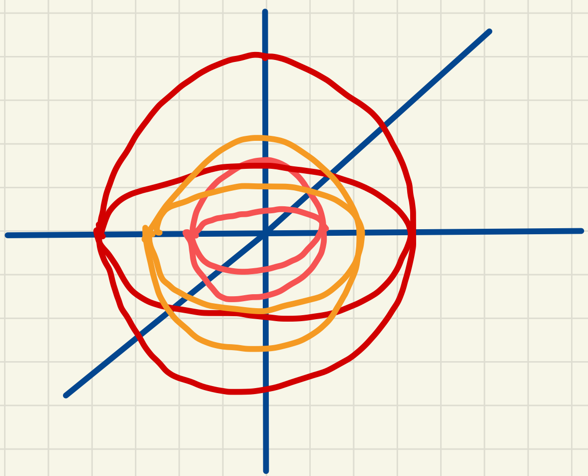
$\vec{u} = \dfrac{\nabla T(1,-1,1)}{|\nabla T(1,-1,1)|} = \dfrac{\langle -3, \frac{9}{2}, -\frac{3}{2} \rangle}{\sqrt{9+\frac{81}{4}+\frac{9}{4}}}$, $-\vec{u} = \dfrac{\langle 3, -\frac{9}{2}, \frac{3}{2} \rangle}{\sqrt{9+\frac{81}{4}+\frac{9}{4}}}$ is the direction of maximal decrease

the rate is $-|\nabla T(1,-1,1)| = -\sqrt{9+\frac{81}{4}+\frac{9}{4}}$

Tangent planes to level surfaces

Recall that a level surface is defined by an equation of the form $F(x,y,z)=k$

Example: $F(x,y,z)=x^2+y^2+z^2$



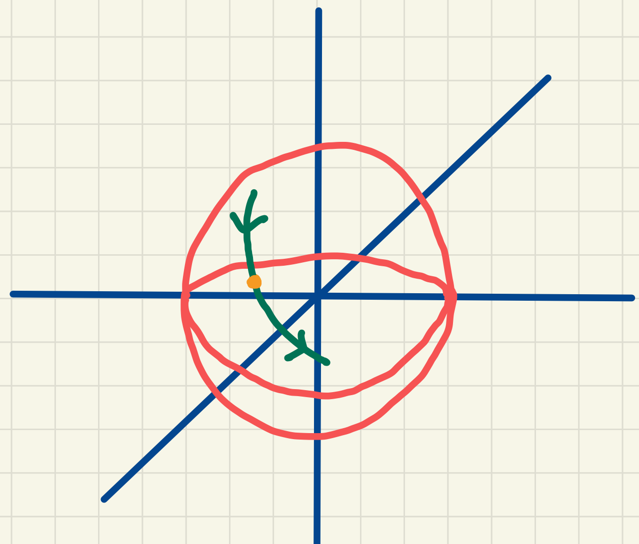
$k=x^2+y^2+z^2$ is a sphere

$F(x,y,z) = k$ defines a level surface

let $P = (x_0, y_0, z_0)$ be a point on this surface

let $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ be a curve on the surface that passes through the point at $t = t_0$

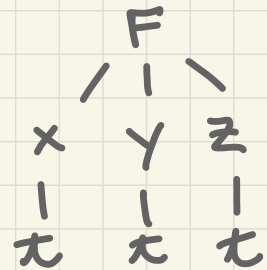
(i.e., $\vec{r}(t_0) = \langle x_0, y_0, z_0 \rangle$)



Takeaway: $F(\vec{r}(t)) = k$ for all t

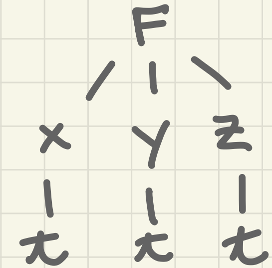
so $\frac{d}{dt}\left[F(\vec{r}(t))\right] = 0$

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$$



Takeaway : $F(\vec{r}(t)) = k$ for all t

so $\frac{d}{dt}\left[F(\vec{r}(t))\right] = 0$



$$\| $$

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$$

$$\| $$

$$\nabla F \cdot \vec{r}\,'(t) = 0$$

in particular, $\nabla F(x_0, y_0, z_0) \cdot \vec{r}\,'(t_0) = 0$

this is true for any differentiable curve $\vec{r}$

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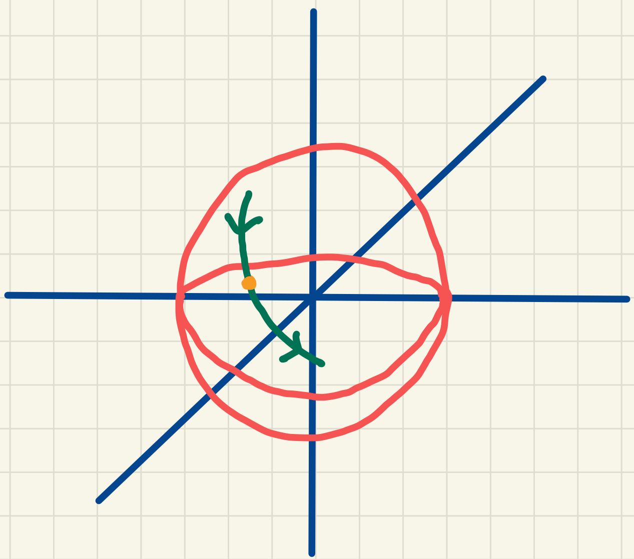
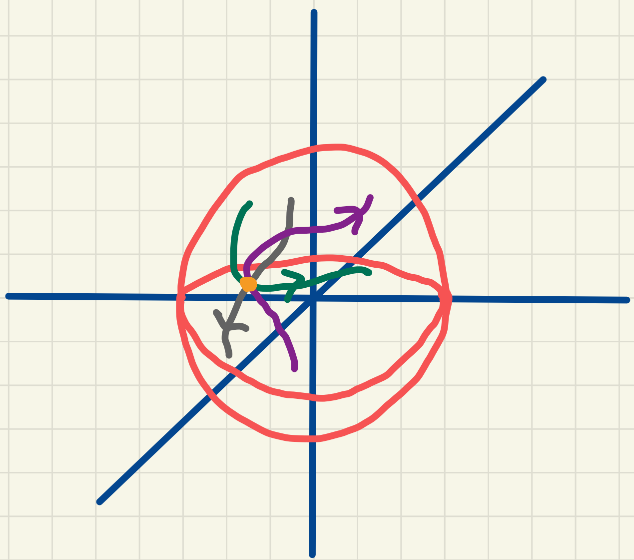




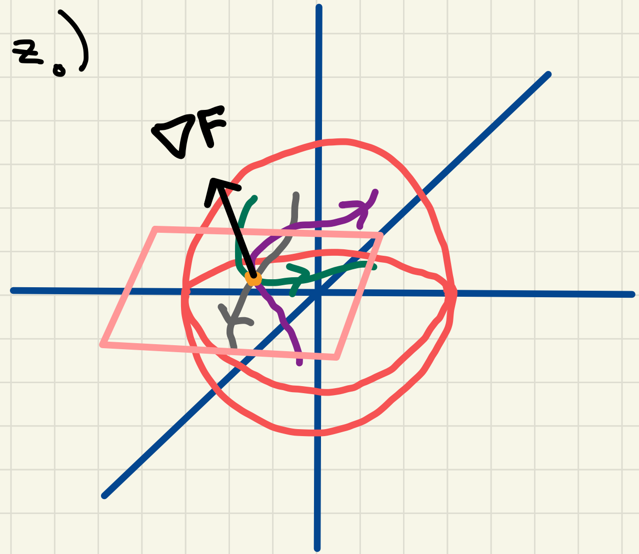
$$\nabla F \cdot \vec{r}\,'(t) = 0$$

in particular, $\nabla F(x_0, y_0, z_0) \cdot \vec{r}\,'(t_0) = 0$

this is true for any differentiable curve $\vec{r}$

Conclusion: $\nabla F(x_0, y_0, z_0)$ is perpendicular to the tangent vector $\vec{r}\,'(t_0)$ of any curve on the level surface that passes through the point $P = (x_0, y_0, z_0)$

if $\nabla F(x_0, y_0, z_0) \neq \vec{0}$, then we define $\nabla F(x_0, y_0, z_0)$ to be the normal vector of the tangent plane to the level surface $F(x,y,z) = k$ at $P = (x_0, y_0, z_0)$

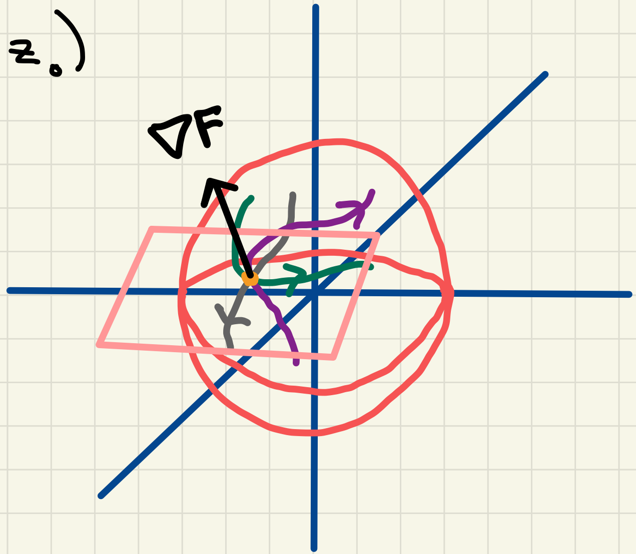


Equation: $\nabla F(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

or $F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$

if $\nabla F(x_0, y_0, z_0) \neq \vec{0}$, then we define $\nabla F(x_0, y_0, z_0)$ to be the parallel vector of the normal line to the level surface $F(x,y,z) = k$ at $P = (x_0, y_0, z_0)$

(perpendicular to the tangent plane)



Equation:

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

Recall: if $z = f(x,y)$ then the tangent plane at

$$\underset{\overset{\downarrow}{=f(x_0,y_0)}}{}$$

(x_0, y_0, z_0) is given by $f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) = z - z_0$

rewrite $z = f(x,y)$ as $F(x,y,z) = f(x,y) - z$

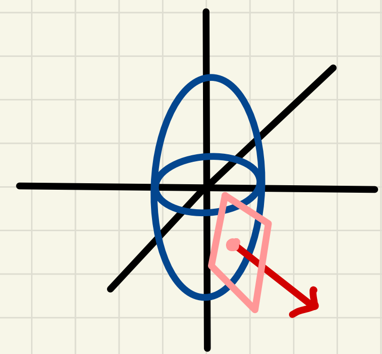
want tangent plane to level surface $F(x,y,z) = 0$

at (x_0,y_0,z_0): $\nabla F(x_0,y_0,z_0) \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0$

$$\nabla F(x_0,y_0,z_0) = \langle f_x(x_0,y_0), f_y(x_0,y_0), -1 \rangle$$

or $f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - (z-z_0) = 0$

Example: find the equations of the tangent plane and normal line at the point $(-2, 1, -3)$ to the ellipsoid $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$



Ans:

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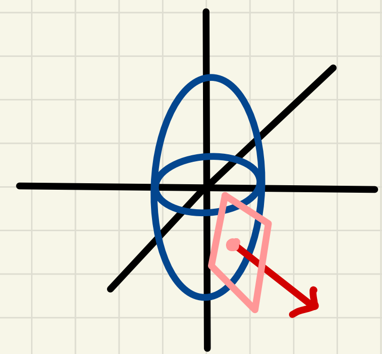


Ans: $F(x,y,z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$

$\nabla F = \langle F_x, F_y, F_z \rangle = \langle \frac{x}{2}, 2y, \frac{2z}{9} \rangle$

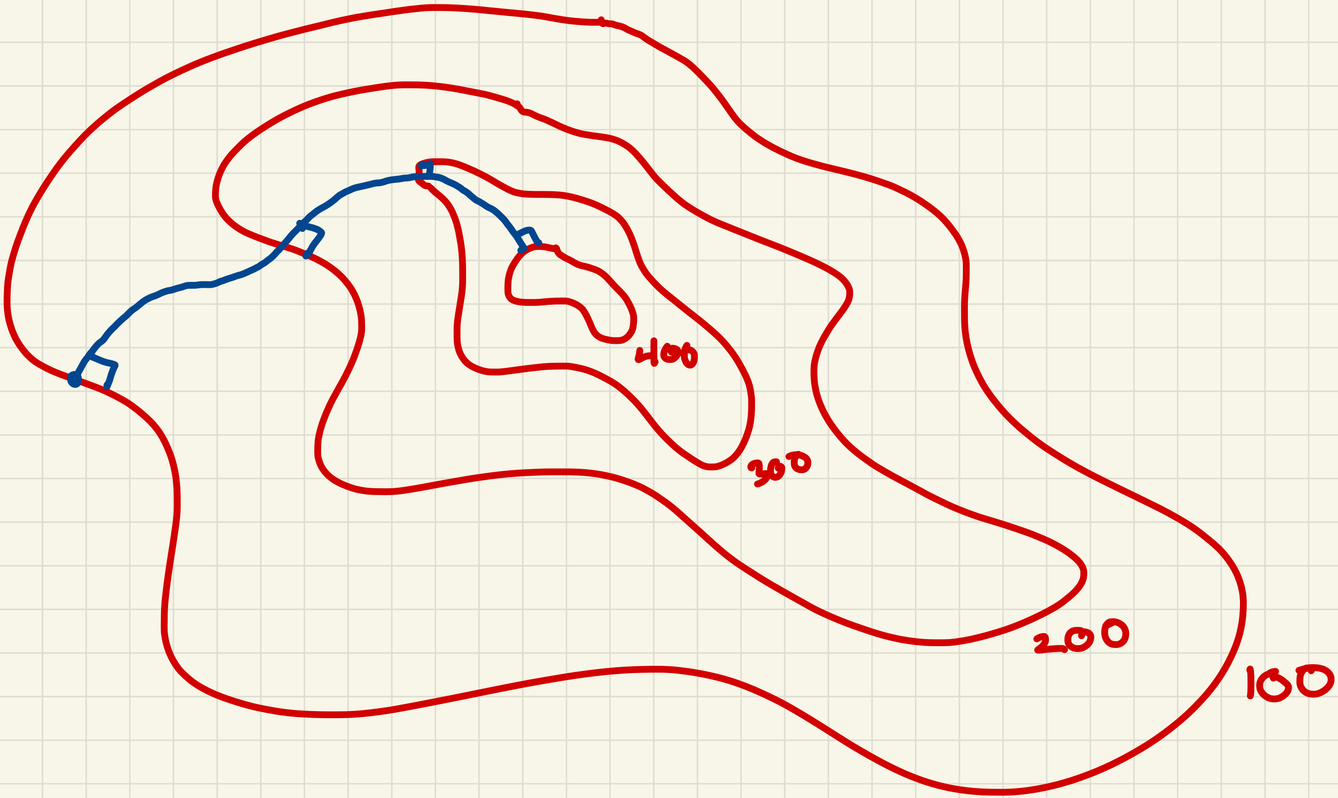
$\nabla F(-2, 1, -3) = \langle -1, 2, \frac{-2}{3} \rangle$

Plane: $-(x+2) + 2(y-1) - \frac{2}{3}(z+3) = 0$

Line: $\frac{x+2}{-1} = \frac{y-1}{2} = \frac{z+3}{-2/3}$

Moral: the gradient tells us a lot

- gradient tells us the direction of fastest increase (opposite direction gives fastest decrease); magnitude of the gradient is the rate
- gradient is orthogonal to level surface



look up "gradient descent" if interested