Iterated integrals

Recall: by definition,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \overbrace{\sum_{i=1}^{n} f(\underbrace{x_i^*}_{\text{sample point}})\,\Delta x}^{\text{Riemann sum}}$$

In practice, we evaluate definite integrals using the fundamental theorem of calculus:

$$\int_a^b f(x)\,dx = F(b) - F(a) \qquad \text{if} \quad F'(x) = f(x)$$

Today: how can we do this for double integrals?

Problem: for a rectangle $R = [a,b] \times [c,d]$,

$$\iint_R f(x,y)\, dA \stackrel{?}{=} F(b,d) - F(a,c) \text{ for}$$

$$F'(x,y) = f(x,y)$$

what are we differentiating with respect to?

Recall: for a function of two variables $f(x,y)$,

we have PARTIAL DERIVATIVES

Today: PARTIAL INTEGRATION

$$\int_c^d f(x,y)\,\underbrace{dy}_{\text{with respect to } y,}$$

with respect to y,
treat x as a constant

(integral analogue of $\frac{d}{dy}$)

$$\int_c^d f(x,y)\,dy$$

with respect to y,
treat x as a constant

(integral analogue of $\frac{d}{dy}$)

this defines a function of x:

$$A(x) = \int_c^d f(x,y)\,dy$$

(Example:

$$A(2) = \int_c^d f(2,y)\,dy$$)

this defines a function of x:

$$A(x) = \int_c^d f(x,y)\,dy$$

now we integrate this function:

$$\int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

"iterated integral"

note: we could have done dx first and then dy to get $\int_c^d \left[\int_a^b f(x,y)\,dx \right] dy$

Example: evaluate $\int_0^3 \int_2^4 x^2 y \, dy \, dx$ and

$$\int_2^4 \int_0^3 x^2 y \, dx \, dy$$

Ans:

Example: evaluate $\displaystyle\int_0^3\int_2^4 x^2y\,dy\,dx$ and

$$\int_2^4\int_0^3 x^2y\,dx\,dy$$

Ans: $\displaystyle\int_0^3\int_2^4 x^2y\,dy\,dx$

$$\left.\frac{x^2y^2}{2}\right|_2^4 = \frac{16x^2}{2} - \frac{4x^2}{2} = 6x^2$$

$$\int_0^3 6x^2\,dx = \left.2x^3\right|_0^3 = 2(3)^3 = 54$$

$\displaystyle\int_2^4\int_0^3 x^2y\,dx\,dy$

$$\left.\frac{x^3y}{3}\right|_0^3 = \frac{3^3y}{3} = 9y$$

$$\int_2^4 9y\,dy = \left.\frac{9y^2}{2}\right|_2^4$$

$$= \frac{9\cdot16}{2} - \frac{9\cdot4}{2} = \frac{9\cdot12}{2} = 54$$

Conclusion: the same!

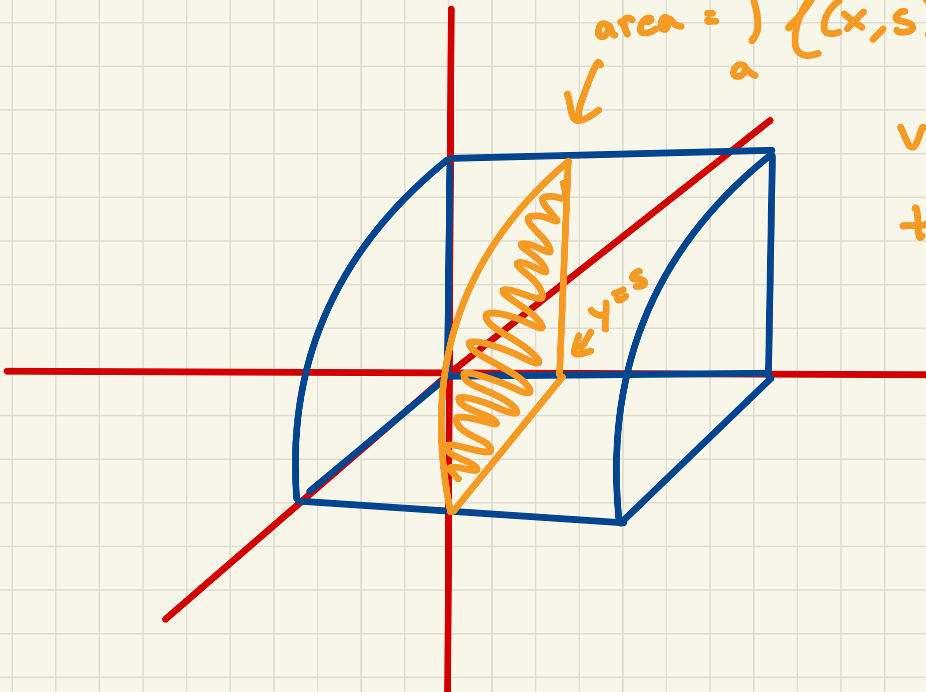
Recall: Clairaut's theorem says order of partial differentiation doesn't matter in certain situations

Fubini's theorem: if f is continuous on the rectangle $R = [a,b] \times [c,d] = \{(x,y) : a \le x \le b,\ c \le y \le d\}$,

then $$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy = \iint_R f(x,y)\,dA$$

Idea for $\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy = \iint_R f(x,y)\,dA$

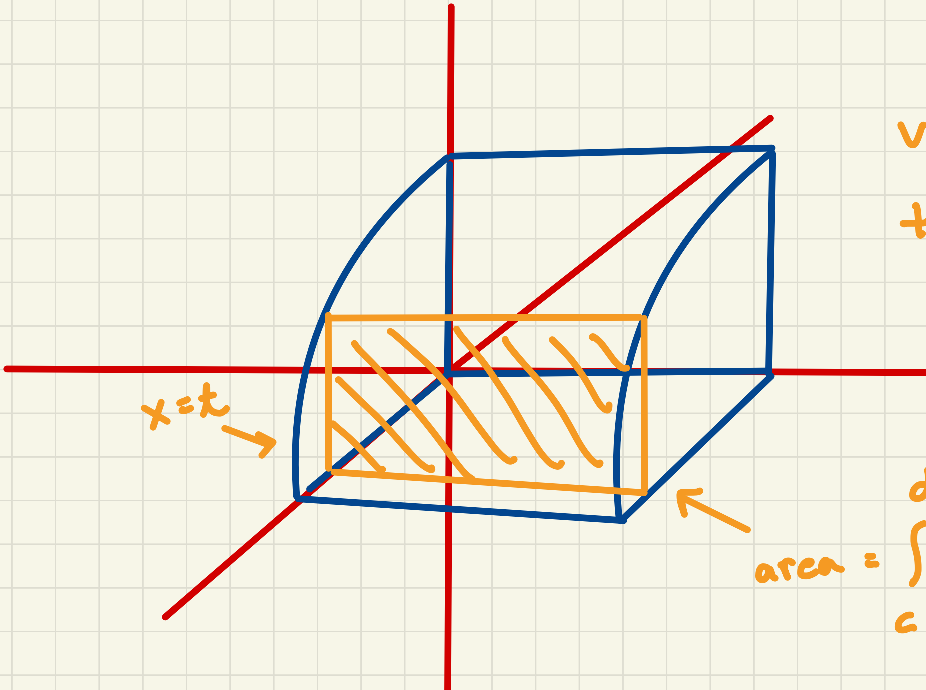
area $= \int_a^b f(x,s)\,dx$



volume "adds" up these slices

$\int_c^d$ slices dy

Idea for $\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy = \iint_R f(x,y)\,dA$



volume "adds" up these slices $\int_a^b \text{slices}\,dx$

$\uparrow$

$\text{area} = \int_c^d f(t,y)\,dy$

Example: compute $\iint_R x-3y^2 \, dA$ for

$R = \{(x,y) : \quad 0 \leq x \leq 2, \quad 1 \leq y \leq 2\} = [0,2] \times [1,2]$

Ans:

Example: compute $\iint_R x-3y^2 \, dA$ for

$R = \{(x,y) : 0 \le x \le 2, \ 1 \le y \le 2\} = [0,2] \times [1,2]$

Ans: $\iint_R x-3y^2 \, dA = \int_0^2 \int_1^2 x-3y^2 \, dy \, dx$

$$xy - y^3 \Big|_1^2 = (2x-8) - (x-1) = x-7$$

$$\int_0^2 x-7 \, dx = \frac{x^2}{2} - 7x \Big|_0^2 = \frac{4}{2} - 14 = -12$$

Try $\int_1^3 \int_0^2 x-3y^2 \, dx \, dy$ at home!

Sometimes order matters from a practical point of view

Example: Compute $\iint_R y\sin(xy)\,dA$ for

$R = [1,2] \times [0,\pi]$

First ans:

Sometimes order matters from a practical point of view

Example: Compute $\iint_R y\sin(xy)\,dA$ for

$R=[1,2]\times[0,\pi]$

First ans: $\int_0^{\pi}\underbrace{\int_1^2 y\sin(xy)\,dx}\,dy$

$$-\cos(xy)\Big|_1^2 = -\cos(2y)+\cos(y)$$

$$\int_0^{\pi} -\cos(2y)+\cos(y)\,dy = -\frac{\sin(2y)}{2}+\sin(y)\Big|_0^{\pi} = 0$$

Example: Compute $\iint_R y\sin(xy)\,dA$ for

$R=[1,2]\times[0,\pi]$

Second ans: $\underbrace{\int_1^2\int_0^\pi y\sin(xy)\,dy}_{\text{integration by parts}}\,dx$

$$\int y\sin(xy)\,dy = -\frac{y\cos(xy)}{x} - \int -\frac{\cos(xy)}{x}\,dy$$

$$= \frac{-y\cos(xy)}{x} + \frac{\sin(xy)}{x^2}$$

Second ans: $\underbrace{\int_1^2 \int_0^\pi y \sin(xy)\, dy dx}_{\text{integration by parts}}$

$$\int_0^\pi y\sin(xy)\, dy = \frac{-y\cos(xy)}{x} + \frac{\sin(xy)}{x^2} \Bigg|_0^\pi$$

$$= \left(\frac{-\pi\cos(\pi x)}{x} + \frac{\sin(\pi x)}{x^2}\right) - (0+0)$$

$$= \frac{-\pi\cos(\pi x)}{x} + \frac{\sin(\pi x)}{x^2}$$

Second ans: $\displaystyle\int_1^2\int_0^\pi y\sin(xy)\,dy\,dx$

$$= \int_1^2 \underbrace{\frac{-\pi\cos(\pi x)}{x}}_{\text{focus on this for now}} + \frac{\sin(\pi x)}{x^2}\,dx$$

$$\int \frac{-\pi\cos(\pi x)}{x}\,dx \underset{\substack{\uparrow\\ \text{integration by parts}}}{=} \frac{-\sin(\pi x)}{x} - \int \frac{\sin(\pi x)}{x^2}\,dx$$

so $$\int \frac{-\pi\cos(\pi x)}{x} + \frac{\sin(\pi x)}{x^2}\,dx = \frac{-\sin(\pi x)}{x}$$

Second ans: $\int_1^2 \int_0^\pi y \sin(xy)\, dy\,dx$

$$= \int_1^2 \frac{-\pi\cos(\pi x)}{x} + \frac{\sin(\pi x)}{x^2}\, dx$$

$$= \left.\frac{-\sin(\pi x)}{x}\right|_1^2 = \frac{-\sin(2\pi)}{2} - \left(-\frac{\sin(\pi)}{1}\right)$$

$$= -0 + 0 = 0$$

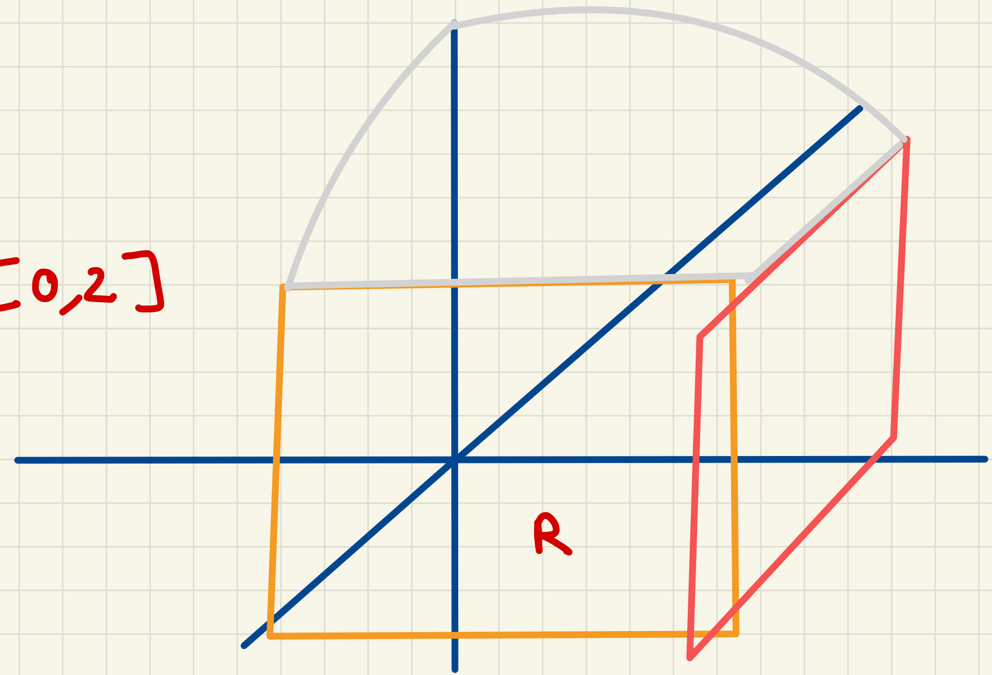
Moral: if one way seems long and tedious, try the other way (be careful about this though!)

Example: find the volume of the solid bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes

Ans:

Example: Find the volume of the solid bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x=2$ and $y=2$, and the three coordinate planes

Ans: $R = [0,2] \times [0,2]$



Example: find the volume of the solid bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x=2$ and $y=2$, and the three coordinate planes

Ans: $\int_0^2 \int_0^2 16 - x^2 - 2y^2 \, dx\, dy = \int_0^2 \left(16x - \frac{x^3}{3} - 2xy^2 \Big|_0^2 \right) dy$

$= \int_0^2 32 - \frac{8}{3} - 4y^2 \, dy = 32y - \frac{8}{3}y - \frac{4y^3}{3} \Big|_0^2 = 64 - \frac{16}{3} - \frac{32}{3}$

$= 48$

Do dydx at home!

A special case: what if our function

$f(x,y) = g(x)h(y)$ factors as a

product of functions $g(x)$ and $h(y)$?

Example: $f(x,y) = x^3y^2 = g(x)h(y)$

for $g(x) = x^3$ and $h(y) = y^2$

Non-example: $f(x,y) = \log(xy) = \log(x) + \log(y)$

If $\quad R = [a,b] \times [c,d] \quad$ and

$$f(x,y) = g(x)h(y)$$

then $$\iint_R f(x,y)\,dA = \int_a^b \int_c^d g(x)h(y)\,dy\,dx$$

$$= \int_a^b g(x) \left(\int_c^d h(y)\,dy \right) dx$$

$$= \int_c^d h(y)\,dy \int_a^b g(x)\,dx$$

Example: if $R = [0, \frac{\pi}{2}] \times [\frac{\pi}{4}, \frac{3\pi}{4}]$,

evaluate $\iint_R \sin(y)\cos(x)\, dA$

Ans:

Example: if $R = [0, \frac{\pi}{2}] \times [\frac{\pi}{4}, \frac{3\pi}{4}]$,

evaluate $\iint_R \sin(y)\cos(x)\, dA$

Ans: $\iint_R \sin(y)\cos(x)\, dA = \int_0^{\frac{\pi}{2}} \cos(x)\, dx \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin(y)\, dy$

$$= \left(\sin(x) \Big|_0^{\frac{\pi}{2}} \right)\left(-\cos(y) \Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \right) = (1-0)\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right)$$

$$= \sqrt{2}$$