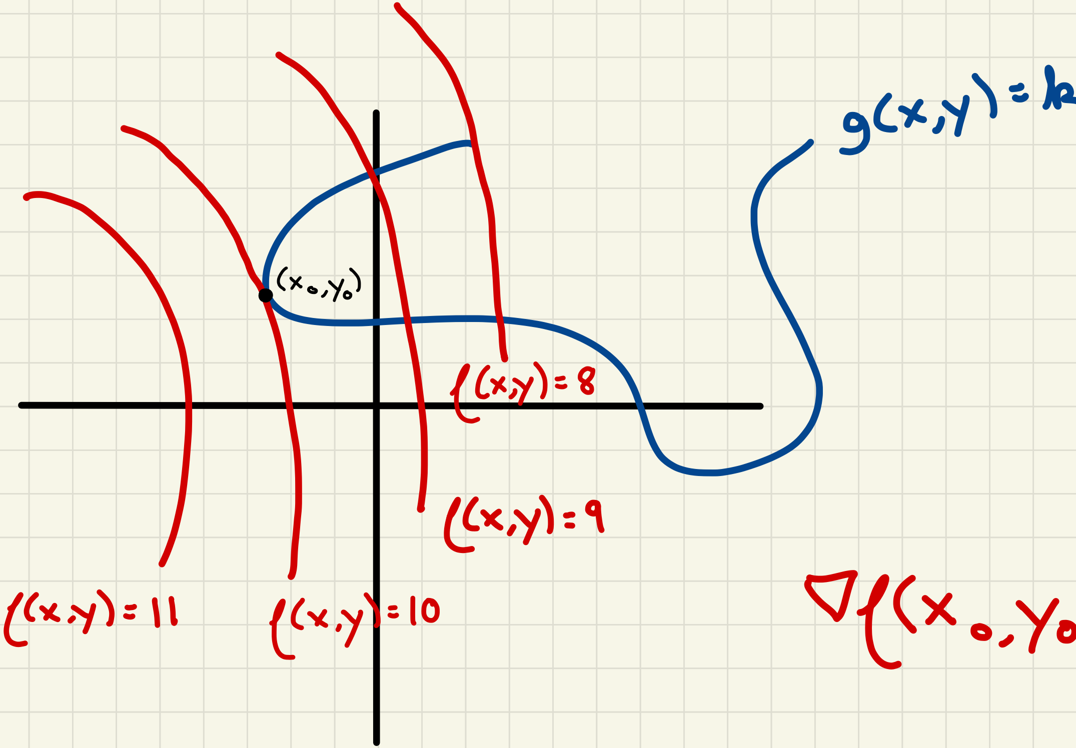
Lagrange multipliers

Last time: finding extrema

Today: finding extrema subject to constraints

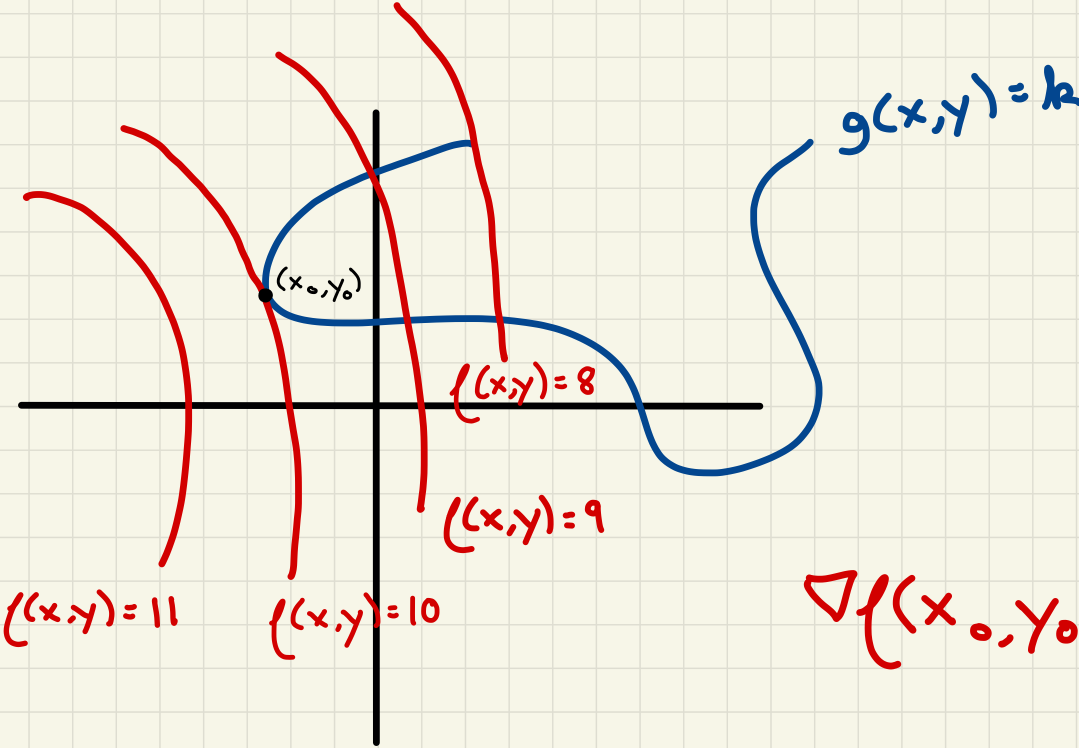
"Lagrange multipliers"

Idea: suppose we want to find extrema of $f(x,y)$ subject to the constraint $g(x,y)=k$



$$\nabla f(x_0,y_0)=\lambda\nabla g(x_0,y_0)$$

Idea: if gradients aren't parallel, we could pick a different level curve for $f(x,y)$ in a way that still intersects $g(x,y)=k$ but increases/decreases our function f

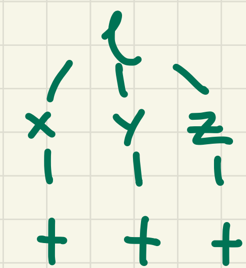


$$\nabla f(x_0,y_0) = \lambda \nabla g(x_0,y_0)$$

Second idea: we should consume to the point at which the relative costs are the same as the relative gains

Same idea for $f(x,y,z)$ subject to $g(x,y,z)=k$

Formally, if $f(x_0,y_0,z_0)$ is an extremum subject to $g(x,y,z)=k$, then let $\vec{r}(t)$ be a curve on $g(x,y,z)=k$ passing through (x_0,y_0,z_0) at $t=t_0$ (i.e. $\vec{r}(t_0)=\langle x_0,y_0,z_0\rangle$)



if $h(t)=f(\vec{r}(t))$, then $h'(t_0)=0$

chain rule says $h'(t)=\nabla f(\vec{r}(t))\cdot\vec{r}\,'(t)$

$$h'(t_0)=\nabla f(x_0,y_0,z_0)\cdot\vec{r}\,'(t_0)=0$$

recall that $\nabla g(x_0,y_0,z_0)\cdot\vec{r}\,'(t_0)=0$

if $\nabla g(x_0,y_0,z_0)\neq\vec{0}$

Lagrange multiplier

so $\nabla f(x_0,y_0,z_0)=\lambda\nabla g(x_0,y_0,z_0)$ for some λ

Method of Lagrange multipliers

To find max and min values of $f(x,y,z)$ subject to $g(x,y,z)=k$ (assuming extrema exist and $\nabla g \neq \vec{0}$ on the surface $g(x,y,z)=k$):

(1) find all points (x,y,z) and values λ such that

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) \quad \text{AND} \quad g(x,y,z)=k$$

(2) evaluate f at the points from (1)

the largest is the max value of f subject to $g(x,y,z)=k$

the smallest is the min value of f subject to $g(x,y,z)=k$

Note: $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$

can be written as the set of equalities

$$f_x = \lambda g_x \quad , \quad f_y = \lambda g_y , \quad \text{and} \quad f_z = \lambda g_z$$

we also still have the constraint $g(x,y,z)=k$

for a function of two variables $f(x,y)$ subject to $g(x,y)=k$. we have

$$f_x = \lambda g_y \quad , \quad f_y = \lambda g_y , \quad g(x,y)=k$$

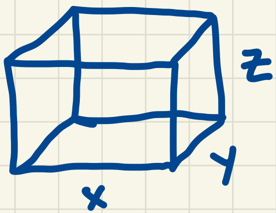
Example: a rectangular box without a lid is to be made from 12 m^2 of cardboard

find the max volume

Ans:

Example: a rectangular box without a lid is to be made from 12 m^2 of cardboard. find the max volume

Ans: $f(x,y,z) = xyz$, $g(x,y,z) = xy + 2xz + 2yz = 12$



$f_x = yz$, $g_x = y + 2z \Rightarrow yz = \lambda(y+2z)$

$f_y = xz$, $g_y = x + 2z \Rightarrow xz = \lambda(x+2z)$

$f_z = xy$, $g_z = 2x + 2y \Rightarrow xy = \lambda(2x+2y)$

$\lambda \neq 0$ since $g(x,y,z) = 0 \neq 12$

$$\lambda = \frac{yz}{y+2z} = \frac{xz}{x+2z} = \frac{xy}{2x+2y}$$

Example: a rectangular box without a lid is to be made from 12 m^2 of cardboard find the max volume

Ans: $f(x,y,z) = xyz$, $g(x,y,z) = xy + 2xz + 2yz = 12$

$$\frac{yz}{y+2z} = \frac{xz}{x+2z} = \frac{xy}{2x+2y}$$

$$= \quad \Rightarrow (2x+2y)\cancel{y}z = x\cancel{y}(y+2z) \Rightarrow 2xz + 2yz = xy + 2xz$$

$$\Rightarrow 2\cancel{y}z = x\cancel{y} \Rightarrow 2z = x$$

$$= \quad \Rightarrow (x+2z)y\cancel{z} = x\cancel{z}(y+2z) \Rightarrow xy + 2yz = xy + 2xz$$

$$\Rightarrow \cancel{2}y\cancel{z} = \cancel{2}x\cancel{z} \Rightarrow x = y$$

Example: a rectangular box without a lid is to be made from 12 m^2 of cardboard

find the max volume

Ans: $f(x,y,z) = xyz$, $g(x,y,z) = xy + 2xz + 2yz = 12$

$2z = x = y$

$x = y$

plug into constraint

$(2z)(2z) + 2(2z)z + 2(2z)z = 12$

$12z^2 = 12$

$z = 1$, $x = y = 2$, max vol is $4\ m^3$

Example: Find the extreme values of the function

$f(x,y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$

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$f(x,y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$

Ans: $f_x = 2x$, $g_x = 2x$ $\Rightarrow$ $2x = \lambda(2x)$

$f_y = 4y$, $g_y = 2y$ $\Rightarrow$ $4y = \lambda(2y)$

$\lambda \neq 0$ since otherwise $x^2 + y^2 = 0 \neq 1$

$x = \lambda x \Rightarrow x(1-\lambda) = 0$

$2y = \lambda y \Rightarrow y(2-\lambda) = 0$

if $1-\lambda = 0$
then $\lambda = 1$
and $y = 0$
and $x = \pm 1$
and $f(\pm 1, 0) = 1$ ← min

if $x = 0$
then $y = \pm 1$
and $f(0, \pm 1) = 2$
max

Example: Find the extreme values of the function

$f(x,y) = x^2 + 2y^2$ on the disk $x^2 + y^2 \leq 1$

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Ans: From 11.7, we compare the extrema we found on the boundary (the circle) to the critical points inside the circle

$$\begin{cases} f_x = 2x \\ f_y = 4y \end{cases} \Rightarrow \text{only critical point is } (0,0)$$

min

$$f(0,\pm 1) = 2, \quad f(\pm 1, 0) = 1, \quad f(0,0) = 0$$

max

Example: find the points on the sphere $x^2+y^2+z^2=4$ that are closest to and farthest away from the point $(3,1,-1)$

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Ans: $F(x,y,z)=\sqrt{(x-3)^2+(y-1)^2+(z+1)^2}$

$f(x,y,z)=(x-3)^2+(y-1)^2+(z+1)^2$

$g(x,y,z)=x^2+y^2+z^2=4$

if $\lambda=0$,
$x=3$, $y=1$, $z=-1$
$\Rightarrow$ not on sphere

$f_x=2(x-3)$, $\quad g_x=2x \quad \Rightarrow \quad 2(x-3)=\lambda(2x)$

$f_y=2(y-1)$, $\quad g_y=2y \quad \Rightarrow \quad 2(y-1)=\lambda(2y)$

$f_z=2(z+1)$, $\quad g_z=2z \quad \Rightarrow \quad 2(z+1)=\lambda(2z)$

Example: Find the points on the sphere $x^2+y^2+z^2=4$ that are closest to and farthest away from the point $(3,1,-1)$

Ans: $F(x,y,z) = \sqrt{(x-3)^2+(y-1)^2+(z+1)^2}$

$f(x,y,z) = (x-3)^2+(y-1)^2+(z+1)^2$

$g(x,y,z) = x^2+y^2+z^2=4$

$2(x-3) = \lambda(2x) \Rightarrow$

$2(y-1) = \lambda(2y) \Rightarrow$

$2(z+1) = \lambda(2z) \Rightarrow$

$\lambda = \frac{x-3}{x} = 1-\frac{3}{x}$

$\lambda = \frac{y-1}{y} = 1-\frac{1}{y}$

$\lambda = \frac{z+1}{z} = 1+\frac{1}{z}$

and give $\frac{3}{x} = \frac{1}{y}$ or $3y = x$

and give $\frac{-3}{x} = \frac{1}{z}$ or $-3z = x$

so $x = 3y = -3z \Rightarrow$ $x = 3y$, $z = -y$

Example: Find the points on the sphere $x^2+y^2+z^2=4$ that are closest to and farthest away from the point $(3,1,-1)$

Ans: $g(x,y,z) = x^2+y^2+z^2=4$

$F(x,y,z) = \sqrt{(x-3)^2+(y-1)^2+(z+1)^2}$

plug into constraint

$f(x,y,z) = (x-3)^2+(y-1)^2+(z+1)^2$

$x=3y$
$z=-y$

$\Rightarrow \quad 9y^2+y^2+y^2=4 \Rightarrow y = \pm\sqrt{\frac{4}{11}} = y = \pm\frac{2}{\sqrt{11}}$

$y = \pm\frac{2}{\sqrt{11}} \Rightarrow z = \mp\frac{2}{\sqrt{11}}, \quad x = \pm\frac{6}{\sqrt{11}}$

farthest

$\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, \frac{-2}{\sqrt{11}}\right)$ and $\left(\frac{-6}{\sqrt{11}}, \frac{-2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$

closest

Two constraints!

$f(x,y,z)$ subject to $g(x,y,z)=k$

$h(x,y,z)=c$

Strategy: look for the points (x_0,y_0,z_0) where

$$\nabla f(x_0,y_0,z_0) = \lambda \nabla g(x_0,y_0,z_0) + \mu \nabla h(x_0,y_0,z_0)$$

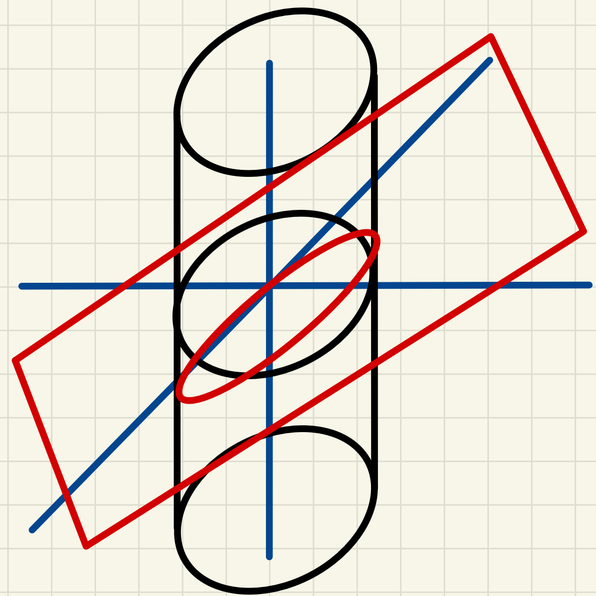
$f_x = \lambda g_x + \mu h_x$, $f_y = \lambda g_y + \mu h_y$, $f_z = \lambda g_z + \mu h_z$

Example: find the extrema of the function

$f(x,y,z) = x + 2y + 3z$ on the curve of

intersection of the plane $x - y + z = 1$ and

the cylinder $x^2 + y^2 = 1$



Ans: $g(x,y,z) = x - y + z = 1$

$h(x,y,z) = x^2 + y^2 = 1$

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Ans: $g(x,y,z) = x - y + z = 1$

$h(x,y,z) = x^2 + y^2 = 1$

$$\begin{cases} f_x = 1, & g_x = 1, & h_x = 2x \\ f_y = 2, & g_y = -1, & h_y = 2y \\ f_z = 3, & g_z = 1, & h_z = 0 \end{cases}$$

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$f(x,y,z) = x + 2y + 3z$ on the curve of

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$f_x = 1, \quad g_x = 1, \quad h_x = 2x$

$f_y = 2, \quad g_y = -1, \quad h_y = 2y$

$f_z = 3, \quad g_z = 1, \quad h_z = 0$

Ans: $g(x,y,z) = x - y + z = 1$

$h(x,y,z) = x^2 + y^2 = 1$

$1 = \lambda + \mu(2x) \Rightarrow -2 = 2\mu x \Rightarrow -1 = \mu x \Rightarrow \mu = \frac{-1}{x}$

$2 = -\lambda + \mu(2y) \Rightarrow 5 = 2\mu y \Rightarrow \frac{5}{2} = \mu y \Rightarrow \mu = \frac{5}{2y}$

$\Rightarrow \frac{-1}{x} = \frac{5}{2y}$

$3 = \lambda + \mu(0) \Rightarrow 3 = \lambda$

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Ans: $g(x,y,z) = x - y + z = 1$

$h(x,y,z) = x^2 + y^2 = 1$

$\frac{-1}{x} = \frac{5}{2y} \Rightarrow -2y = 5x \Rightarrow x = \frac{-2}{5}y$

plug in

$\frac{4}{25}y^2 + y^2 = 1 \Rightarrow y^2 = \frac{25}{29}$

plug in

plug in

$\Rightarrow y = \pm\frac{5}{\sqrt{29}} \Rightarrow x = \mp\frac{2}{\sqrt{29}} \Rightarrow z = 1 \pm \frac{7}{\sqrt{29}}$

Example: find the extrema of the function

$f(x,y,z) = x + 2y + 3z$ on the curve of

intersection of the plane $x - y + z = 1$ and

the cylinder $x^2 + y^2 = 1$

Ans: $y = \pm \frac{5}{\sqrt{29}} \Rightarrow x = \mp \frac{2}{\sqrt{29}} \Rightarrow z = 1 \pm \frac{7}{\sqrt{29}}$

$f\left(-\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}, 1 + \frac{7}{\sqrt{29}}\right) = 3 + \frac{29}{\sqrt{29}} = 3 + \sqrt{29}$ ← max

$f\left(\frac{2}{\sqrt{29}}, \frac{-5}{\sqrt{29}}, 1 - \frac{7}{\sqrt{29}}\right) = 3 - \frac{29}{\sqrt{29}} = 3 - \sqrt{29}$ ← min