Vectors

A vector is a quantity that has both magnitude and direction

Examples

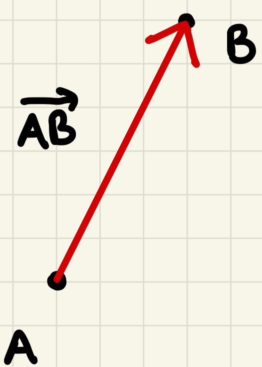
displacement : moving from point A to point B

velocity : moving at a certain speed in a certain direction

force : exertion in a particular direction

Notation: $\vec{v}$

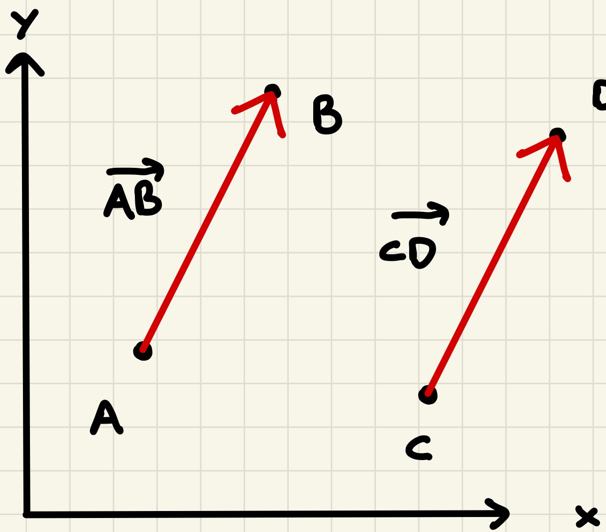
if A, B are points, then $\overrightarrow{AB}$ is the displacement vector of an object starting at point A and moving to point B



A is the initial point (or tail)

B is the terminal point (or tip)

Vectors are equivalent if they have the same magnitude and direction



$\overrightarrow{AB} = \overrightarrow{CD}$

Special vector:

$\vec{0}$ is the zero vector,

has length 0

no direction needed

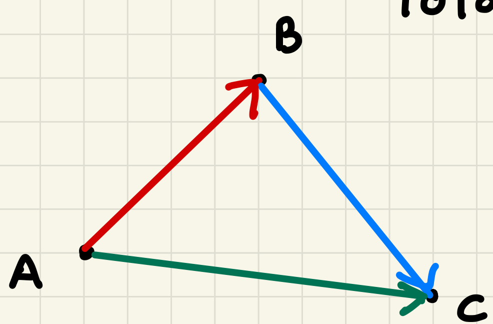
For example, $\overrightarrow{AA} = \vec{0}$

Combining vectors

$A \to B$ displacement is $\overrightarrow{AB}$

$B \to C$ displacement is $\overrightarrow{BC}$

total displacement is $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$



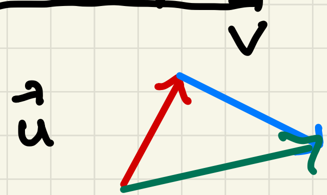
$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

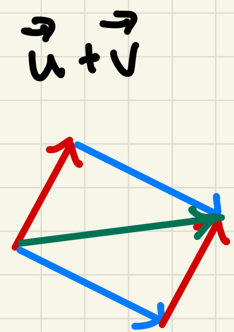
Addition of vectors: suppose $\vec{u}, \vec{v}$ are vectors

place the tail of $\vec{v}$ at the tip of $\vec{u}$

$\vec{u}+\vec{v}$ is the vector from the tail of $\vec{u}$ to the tip of $\vec{v}$

Example

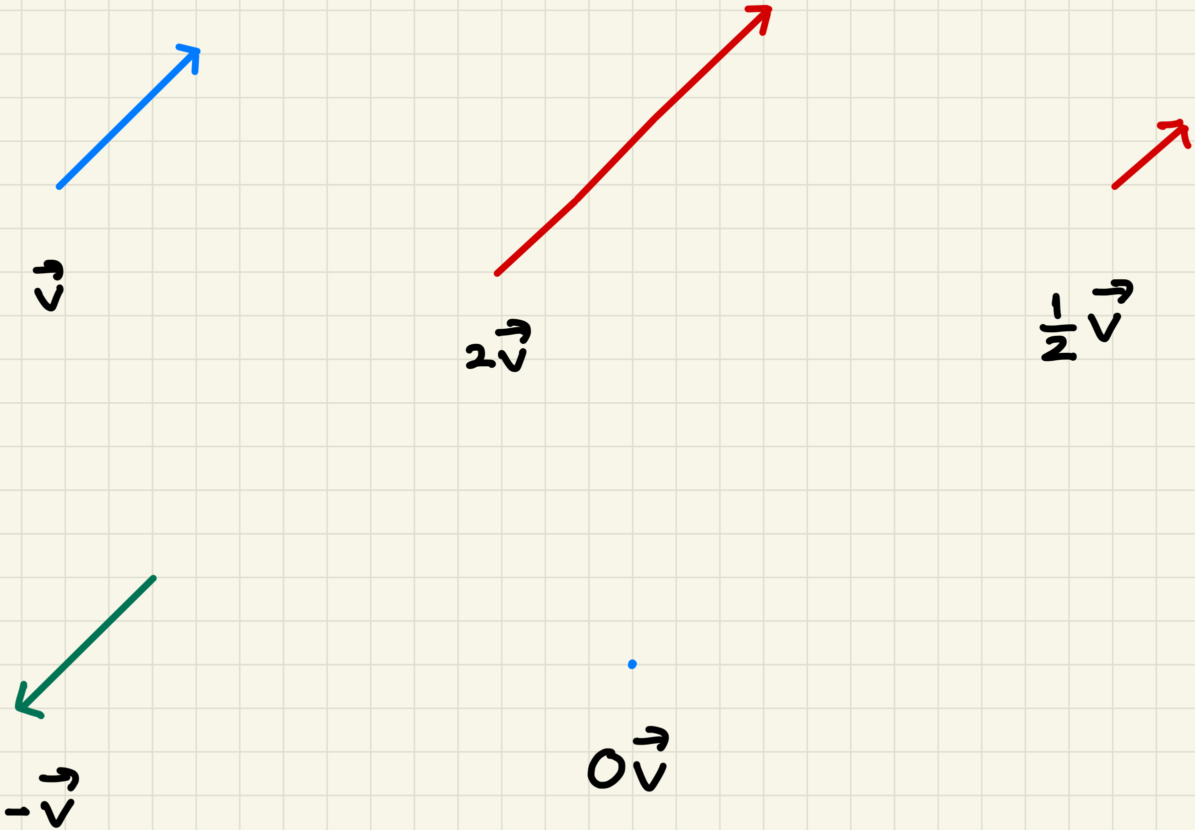




Scalar multiplication : if $c \in \mathbb{R}$ and $\vec{v}$ is a vector then $c\vec{v}$ is the vector whose length is $|c|$ times the length of $\vec{v}$

and in :

- the same direction if $c > 0$
- the opposite direction if $c < 0$
- no particular direction if $c = 0$ since then $c\vec{v} = \vec{0}$


$\vec{v}$
$2\vec{v}$
$\frac{1}{2}\vec{v}$
$-\vec{v}$
$0\vec{v}$

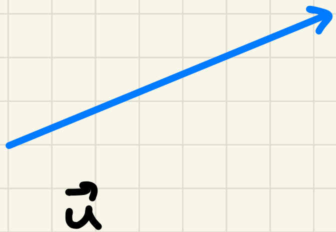
Two nonzero vectors are parallel if they are scalar multiples of each other.

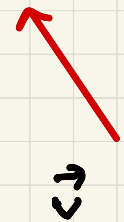
Example if $\vec{v} \neq \vec{0}$, then $\vec{v}$ and $-\vec{v}$ are parallel

$-\vec{v}$ is the negative of $\vec{v}$

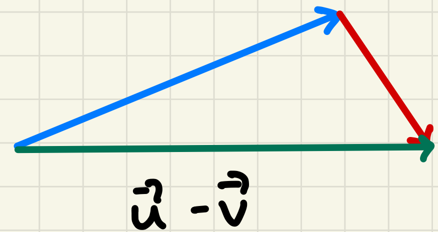
We can also subtract vectors:

$$\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$$

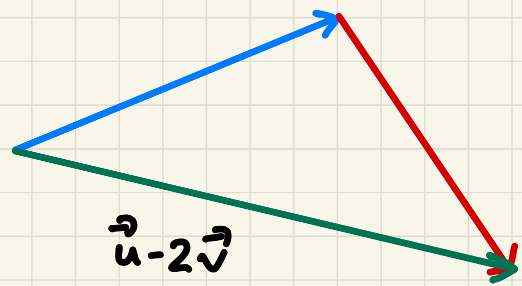




$$\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$$

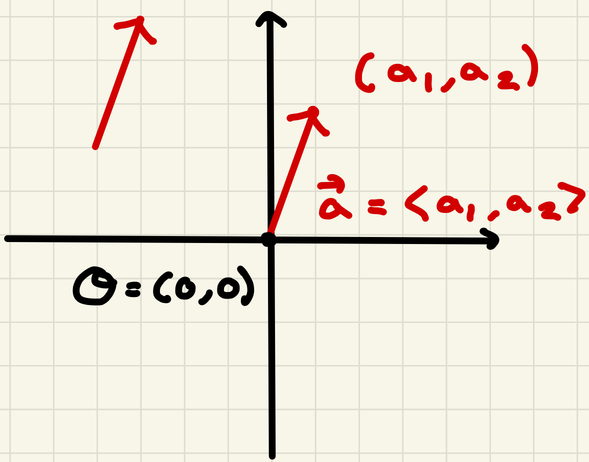


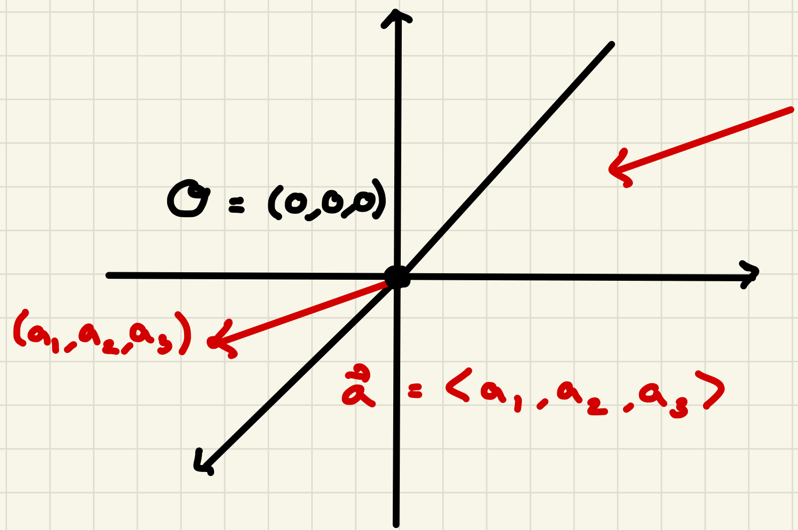
$$\vec{u} - 2\vec{v} = \vec{u} + (-2\vec{v})$$



Components

since vectors are equivalent if they have the same magnitude and direction, we can place the tail at the origin of our coordinate system





then the vector is completely determined by the coordinates of the terminal point

Important : $(a_1, a_2, a_3) \neq \langle a_1, a_2, a_3 \rangle$

$\swarrow$ point in $\mathbb{R}^3$ $\qquad$ $\downarrow$ vector

if $P = (a_1, a_2, a_3)$ and $\vec{u} = \langle a_1, a_2, a_3 \rangle$,

then $\overrightarrow{OP} = \vec{u}$

$\overrightarrow{OP}$ is the position vector of the point P

but there are many representations of the same vector

Suppose A, B are points in $\mathbb{R}^3$ with coordinates $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$

then $\vec{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$

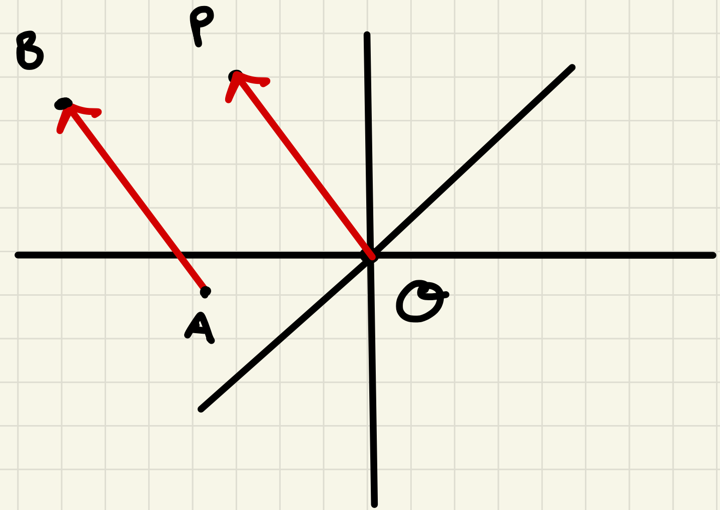
<u>Example</u>

$P = (a_1, a_2, a_3)$

$A = (x_1, y_1, z_1)$

$B = (x_1 + a_1, y_1 + a_2, z_1 + a_3)$

then $\vec{OP} = \vec{AB}$



Concrete example: $A = (0, 1, -2)$

$B = (3, 5, 7)$

$$\vec{AB} = \langle 3-0,\ 5-1,\ 7-(-2) \rangle$$

$$= \langle 3,\ 4,\ 9 \rangle$$

Magnitude (length)

the length of a vector $\vec{a} = \langle a_1, a_2 \rangle$ in $\mathbb{R}^2$

is $|\vec{a}| = \sqrt{a_1^2 + a_2^2}$

the length of a vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$ in $\mathbb{R}^3$

is $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

$\underline{\text{Moral}}$: coordinates make vector operations concretely computable

in $\mathbb{R}^3$: if $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$

and $c \in \mathbb{R}$

then

$$\vec{a} + \vec{b} = \langle a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3 \rangle$$

$$\vec{a} - \vec{b} = \langle a_1 - b_1,\ a_2 - b_2,\ a_3 - b_3 \rangle$$

$$c\vec{a} = \langle ca_1,\ ca_2,\ ca_3 \rangle$$

Examples: $\vec{a} = \langle -1, 2, 4 \rangle$ and $\vec{b} = \langle 3, 0, 1 \rangle$

then $|\vec{a}| = \sqrt{(-1)^2 + (2)^2 + (4)^2} = \sqrt{21}$

$$2\vec{a} - 3\vec{b} = \langle -2, 4, 8 \rangle - \langle 9, 0, 3 \rangle$$

$$= \langle -11, 4, 5 \rangle$$

vectors in $\mathbb{R}^2$, in $\mathbb{R}^3$, in $\mathbb{R}^n$

$\vec{a} = \langle a_1, \dots, a_n \rangle$

addition, subtraction, scalar multiplication

are still done componentwise

later: length?

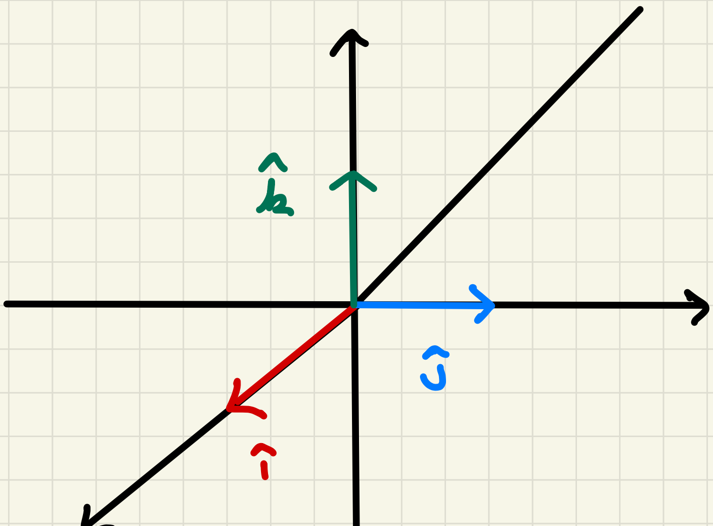
Properties of vector operations in $\mathbb{R}^n$

1. $\vec{a}+\vec{b}=\vec{b}+\vec{a}$
2. $\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}$
3. $\vec{a}+\vec{0}=\vec{a}$
4. $\vec{a}+(-\vec{a})=\vec{0}$
5. $c(\vec{a}+\vec{b})=c\vec{a}+c\vec{b}$
6. $(c+d)\vec{a}=c\vec{a}+d\vec{a}$
7. $(cd)\vec{a}=c(d\vec{a})$
8. $1\vec{a}=\vec{a}$

Special vectors in $\mathbb{R}^3$:

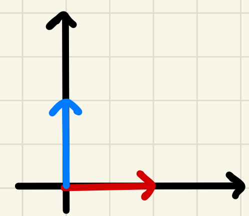
$$\hat{\imath} = \langle 1, 0, 0 \rangle$$
$$\hat{\jmath} = \langle 0, 1, 0 \rangle$$
$$\hat{k} = \langle 0, 0, 1 \rangle$$

be careful in WebAssign (use the symbols)



in $\mathbb{R}^2$, $\hat{\imath} = \langle 1, 0 \rangle$

$\hat{\jmath} = \langle 0, 1 \rangle$



$\hat{\imath}, \hat{\jmath}, \hat{k}$ are called the standard basis vectors

can be used to express any vector in $\mathbb{R}^3$:

$$\langle \pi, 2, -\sqrt{5} \rangle = \pi \hat{\imath} + 2\hat{\jmath} - \sqrt{5}\,\hat{k}$$

$$\langle a_1, a_2, a_3 \rangle = a_1 \hat{\imath} + a_2 \hat{\jmath} + a_3 \hat{k}$$

Unit vectors

Motivation: you just care about the direction

a unit vector is a vector of length 1

Exercise: if $c \in \mathbb{R}$, then $|c\vec{v}| = |c|\,|\vec{v}|$

If $\vec{v} \neq \vec{0}$, then the unit vector in the same direction is $\frac{1}{|\vec{v}|}\vec{v} = \frac{\vec{v}}{|\vec{v}|}$

Examples

$\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ are all unit vectors

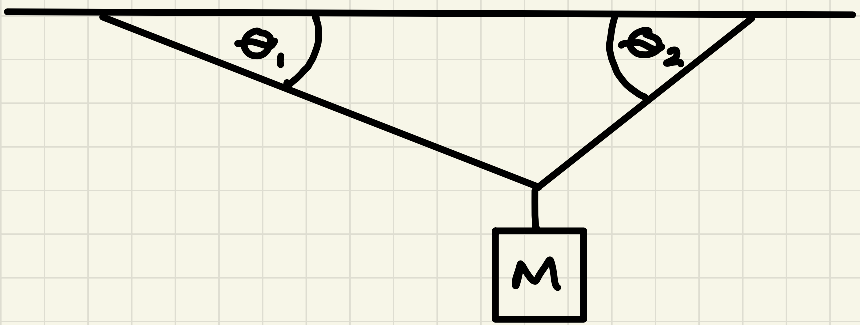
the unit vector in the direction of

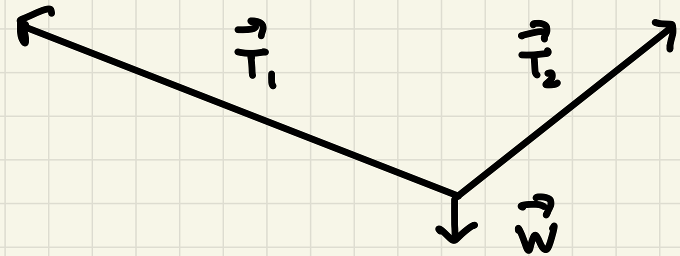
$\hat{\imath} - 2\hat{\jmath} + 3\hat{k} = \langle 1, -2, 3 \rangle$?

$|\langle 1, -2, 3 \rangle| = \sqrt{1 + 4 + 9} = \sqrt{11}$,

so $\dfrac{\hat{\imath} - 2\hat{\jmath} + 3\hat{k}}{\sqrt{11}} = \left\langle \dfrac{1}{\sqrt{11}}, \dfrac{-2}{\sqrt{11}}, \dfrac{3}{\sqrt{11}} \right\rangle$

Applications





$$\vec{T}_1 + \vec{T}_2 + \vec{W} = \vec{0} \quad \text{and} \quad \vec{W} = \langle 0, -M \rangle$$

$$\vec{T}_1 = \langle -\cos(\theta_1)|\vec{T}_1|, \ \sin(\theta_1)|\vec{T}_1| \rangle$$

$$\vec{T}_2 = \langle \cos(\theta_2)|\vec{T}_2|, \ \sin(\theta_2)|\vec{T}_2| \rangle$$

$$\Rightarrow \quad -\cos(\theta_1)|\vec{T}_1| + \cos(\theta_2)|\vec{T}_2| = 0$$

$$\sin(\theta_1)|\vec{T}_1| + \sin(\theta_2)|\vec{T}_2| - M = 0$$

given $\theta_1, \theta_2,$ and M, can solve eq