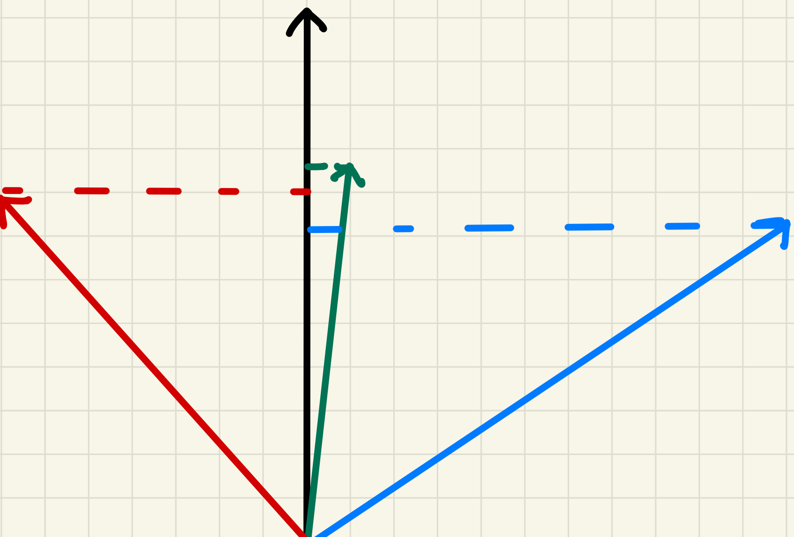
Dot product

Last time: vectors

This time: relationship between vectors

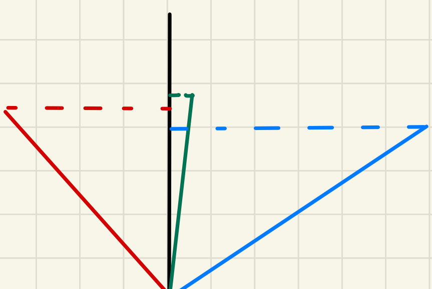
For example, suppose we have two vectors $\vec{v}$ and $\vec{w}$. We know that $\vec{v}$ and $c\vec{v}$ point in the same direction if $c>0$. How do we quantify the extent to which $\vec{v}$ and $\vec{w}$ point in the same direction? Perpendicular?

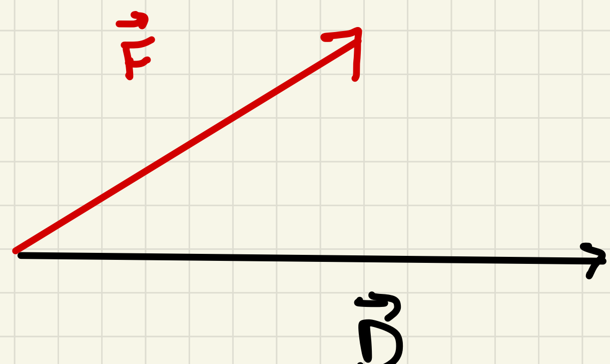
Motivation: consider tossing an object and measuring the distance it has traveled in a particular direction



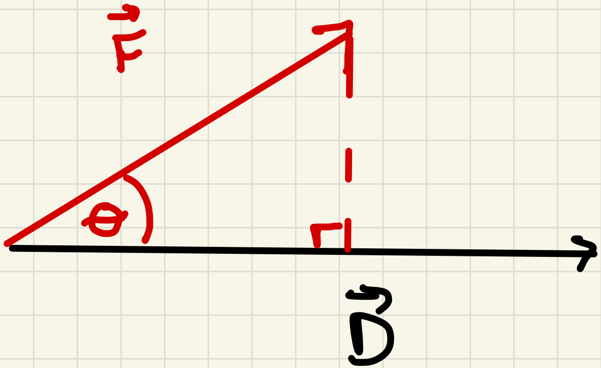
Classic example: the work W done by a constant force F in moving an object through a distance d is $W = Fd$

Caveat: formula only applies when the force is directed along the line of motion





$W \neq |\vec{F}||\vec{D}|$ in general

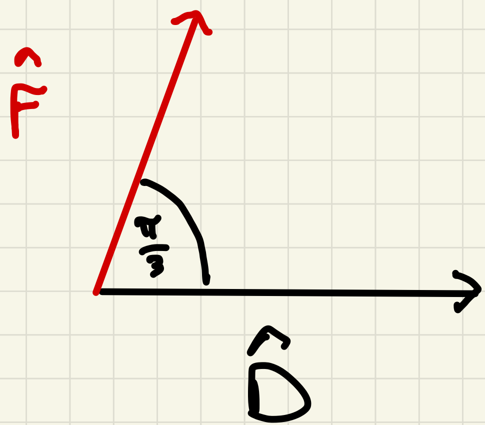


$$W = |\vec{D}||\vec{F}|\cos(\theta)$$

The dot product of two vectors $\vec{a}$ and $\vec{b}$ is $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos(\theta)$, where $\theta \in [0, \pi]$ is the angle between them

Example: A wagon is pulled a distance of 50 m along a horizontal path by a constant force of 70 N. The handle of the wagon is held at angle $\frac{\pi}{3}$

Find the work done by the force.



$$W = 50\,m \cdot 70\,N \cdot \cos\left(\frac{\pi}{3}\right)$$

$$= 1750\ N \cdot m = 1750\ J$$

Two vectors $\vec{a}, \vec{b}$ are orthogonal if and only if $\vec{a} \cdot \vec{b} = 0$ $\quad (\vec{a} \perp \vec{b})$

Note: $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos(\theta)$

assume $\vec{a}, \vec{b} \neq \vec{0}$

then $|\vec{a}|, |\vec{b}| > 0$

so, $\vec{a} \cdot \vec{b} = 0$ forces $\cos(\theta) = 0$,

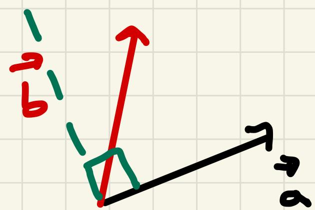
which means $\theta = \frac{\pi}{2}$

In general, if $\vec{a}, \vec{b} \neq \vec{0}$

$$\boxed{\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos(\theta)}$$

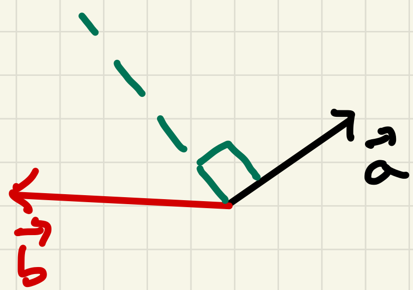
$\vec{a} \cdot \vec{b} > 0$ means $\theta \in [0, \frac{\pi}{2})$

acute angle



$\vec{a} \cdot \vec{b} < 0$ means $\theta \in (\frac{\pi}{2}, \pi]$

obtuse angle

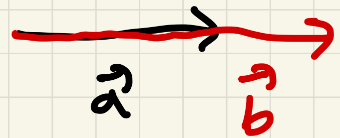


$$\boxed{\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos(\theta)}$$

Extreme cases:

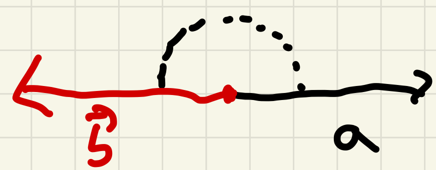
if $\theta = 0$, $\quad \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|$

what does $\theta = 0$ mean?



if $\theta = \pi$, $\quad \vec{a}\cdot\vec{b} = -|\vec{a}||\vec{b}|$

what does $\theta = \pi$ mean?

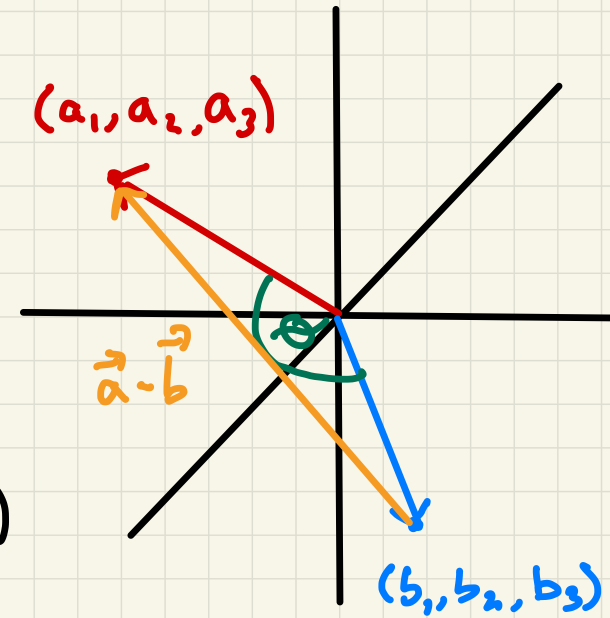


Components

Computing dot products w/ components:

if $\vec{a} = \langle a_1, a_2, a_3 \rangle$

$\vec{b} = \langle b_1, b_2, b_3 \rangle$



Law of cosines

$$|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - \overbrace{2|\vec{a}||\vec{b}|\cos(\theta)}^{\vec{a} \cdot \vec{b}}$$

so,

$$\vec{a} \cdot \vec{b} = \frac{1}{2}\left(|\vec{a}|^2 + |\vec{b}|^2 - |\vec{a} - \vec{b}|^2\right) = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\boxed{\begin{array}{l}\text{If } \vec{a} = \langle a_1, a_2, a_3 \rangle \text{ and } \vec{b} = \langle b_1, b_2, b_3 \rangle \\ \text{then } \vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3\end{array}}$$

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos(\theta)$$

Example: What is the angle $\theta \in [0, \pi]$ between $\langle 1, 1, 0 \rangle$ and $\langle 0, 1, -1 \rangle$?

Ans: $|\langle 1, 1, 0 \rangle| = \sqrt{2}$ $\qquad \langle 1,1,0 \rangle \cdot \langle 0, 1, -1 \rangle = 1$

$|\langle 0, 1, -1 \rangle| = \sqrt{2}$

$$1 = \langle 1, 1, 0 \rangle \cdot \langle 0, 1, -1 \rangle = \sqrt{2} \cdot \sqrt{2} \cdot \cos(\theta)$$

$$\cos(\theta) = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

Example: A force $\vec{F} = \langle 4,3,2 \rangle$ moves a particle from $P=(1,1,1)$ to $Q=(5,2,3)$. Find the work done by the force. (assume distance is in meters, force in N)

Ans: displacement $\vec{D} = \overrightarrow{PQ} = \langle 4,1,2 \rangle$

$$\vec{F} \cdot \vec{D} = 4 \cdot 4 + 3 \cdot 1 + 2 \cdot 2 = 23 \text{ J}$$

Example: find a unit vector that is orthogonal to both $\hat{i}+\hat{j}$ and $\hat{j}+\hat{k}$

Ans: if $\vec{a} = \langle a_1, a_2, a_3 \rangle$ is s.t.

$\vec{a} \perp \underset{\langle 1,1,0\rangle}{(\hat{i}+\hat{j})}$, then $a_1 + a_2 = 0$

$\vec{a} \perp \underset{\langle 0,1,1\rangle}{(\hat{j}+\hat{k})}$, then $a_2 + a_3 = 0$

so, $a_1 = -a_2 = a_3$

need $|\vec{a}| = 1$, so $a_1^2 + a_2^2 + a_3^2 = 3a_1^2 = 1$ $\quad a_1 = \pm\frac{1}{\sqrt{3}}$

can choose $\hat{a} = \left\langle \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$

if $\vec{a}, \vec{b}, \vec{c}$ are three-dimensional vectors and $k \in \mathbb{R}$, then

1. $\vec{a} \cdot \vec{a} = |a|^2$

2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

4. $(k\vec{a}) \cdot \vec{b} = k(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (k\vec{b})$

5. $\vec{0} \cdot \vec{a} = 0$

5. $\vec{0} \cdot \vec{a}$

$= \langle 0,0,0 \rangle \cdot \langle a_1, a_2, a_3 \rangle$

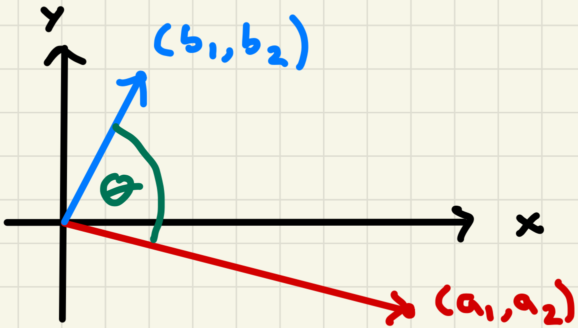
$= 0 \cdot a_1 + 0 \cdot a_2 + 0 \cdot a_3$

$= 0 + 0 + 0 = 0$

~~6.~~ $(\vec{a} \cdot \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \cdot \vec{c})$

Try proving these by writing out the definitions

Note : dot product makes sense in $\mathbb{R}^2$

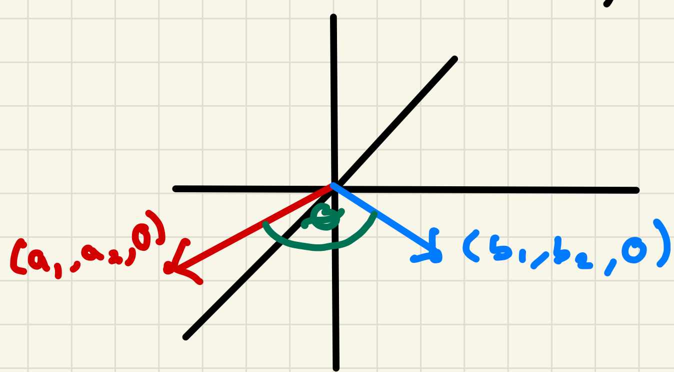


$\vec{a} = \langle a_1, a_2 \rangle$

$\vec{b} = \langle b_1, b_2 \rangle$

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos(\theta)$$
$$= a_1 b_1 + a_2 b_2$$

why? think of the xy-plane in $\mathbb{R}^3$



Show that if $\vec{u}+\vec{v} \perp \vec{u}-\vec{v}$

then $|\vec{u}| = |\vec{v}|$.

Ans: $(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) = 0$ Know this by assumption

LHS $= \vec{u}\cdot(\vec{u}-\vec{v}) + \vec{v}\cdot(\vec{u}-\vec{v})$

$= \vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} - \vec{v}\cdot\vec{v}$

$= |\vec{u}|^2 - \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{v} - |\vec{v}|^2$

$= |\vec{u}|^2 - |\vec{v}|^2 = 0 \Rightarrow |\vec{u}|^2 = |\vec{v}|^2$

$\Rightarrow |\vec{u}| = |\vec{v}|$

Suppose that we are given $\vec{u}, \vec{v}$.

If $\vec{u}\cdot\vec{w} = \vec{v}\cdot\vec{w}$ for any $\vec{w}$,

then $\vec{u} = \vec{v}$

$\vec{u}\cdot\vec{w} = \vec{v}\cdot\vec{w} \Rightarrow \vec{u}\cdot\vec{w} - \vec{v}\cdot\vec{w} = 0$

$\Rightarrow (\vec{u}-\vec{v})\cdot\vec{w} = 0$

let $\vec{w} = \vec{u}-\vec{v}$

Moral: think of this as testing in a certain direction if we test in every direction and get the same result, we must have the same thing to begin with

$(\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) = 0$

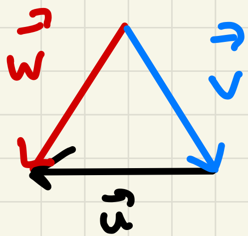
$|\vec{u}-\vec{v}|^2 = 0$

$\vec{u}-\vec{v} = \vec{0}$

$\vec{u} = \vec{v}$

Note: testing in one direction is not enough (e.g., $\hat{\imath}\cdot\hat{k} = \hat{\jmath}\cdot\hat{k}$ but $\hat{\imath} \neq \hat{\jmath}$)

Suppose the vectors $\vec{u}, \vec{v}, \vec{w}$ are unit vectors that can be arranged into an equilateral triangle



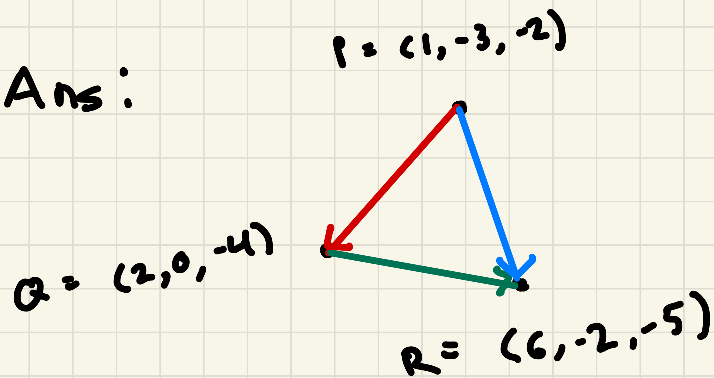
what is $\vec{u} \cdot \vec{w} = |\vec{u}||\vec{w}| \cos(\theta) = (1)(1)\cos(\frac{\pi}{3}) = \frac{1}{2}$

$\vec{v} \cdot \vec{w} = (1)(1)\cos(\frac{\pi}{3}) = \frac{1}{2}$

$\vec{u} \cdot \vec{v} = (1)(1)\cos(\frac{2\pi}{3}) = \frac{-1}{2}$

Use vectors to decide whether or not the triangle with vertices $(1,-3,-2)$, $(2,0,-4)$, and $(6,-2,-5)$ is a right triangle

Ans:



$\overrightarrow{PQ} = \langle 1, 3, -2 \rangle$

$\overrightarrow{QR} = \langle 4, -2, -1 \rangle$

$\overrightarrow{PR} = \langle 5, 1, -3 \rangle$

$\overrightarrow{PQ} \cdot \overrightarrow{QR} = 4 - 6 + 2 = 0$

For which values of b are the vectors $\langle -6, b, 2\rangle$ and $\langle b, b^2, b\rangle$ orthogonal?

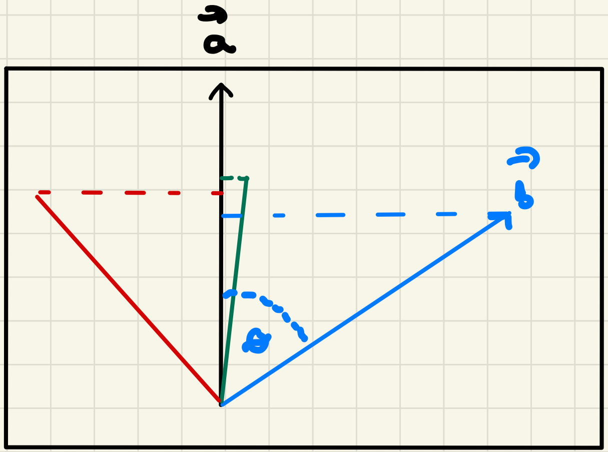
Ans: $\langle -6, b, 2\rangle \cdot \langle b, b^2, b\rangle = -6b + b^3 + 2b$

$$= b^3 - 4b$$

$$= b(b^2 - 4)$$

$$b(b^2-4) = 0 \iff b = 0, \pm 2$$

Projections



$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos(\theta)$$

$$|\vec{b}|\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \frac{\vec{a}}{|\vec{a}|} \cdot \vec{b}$$

this is what we want

the scalar projection of $\vec{b}$ onto $\vec{a}$

is $\operatorname{comp}_{\vec{a}}(\vec{b}) = \dfrac{\vec{a}}{|\vec{a}|} \cdot \vec{b}$

informally, how far does $\vec{b}$ travel in the direction of $\vec{a}$

(note : can be negative)

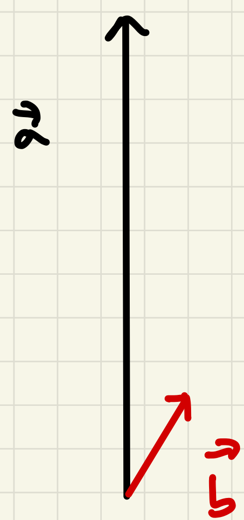
$$\text{comp}_{\vec{a}}(\vec{b}) = \frac{\vec{a}}{|\vec{a}|} \cdot \vec{b}$$

informally, how far does $\vec{b}$ travel in the direction of $\vec{a}$

Recall, $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

In general, $\text{comp}_{\vec{a}}(\vec{b}) \neq \text{comp}_{\vec{b}}(\vec{a})$

Example:

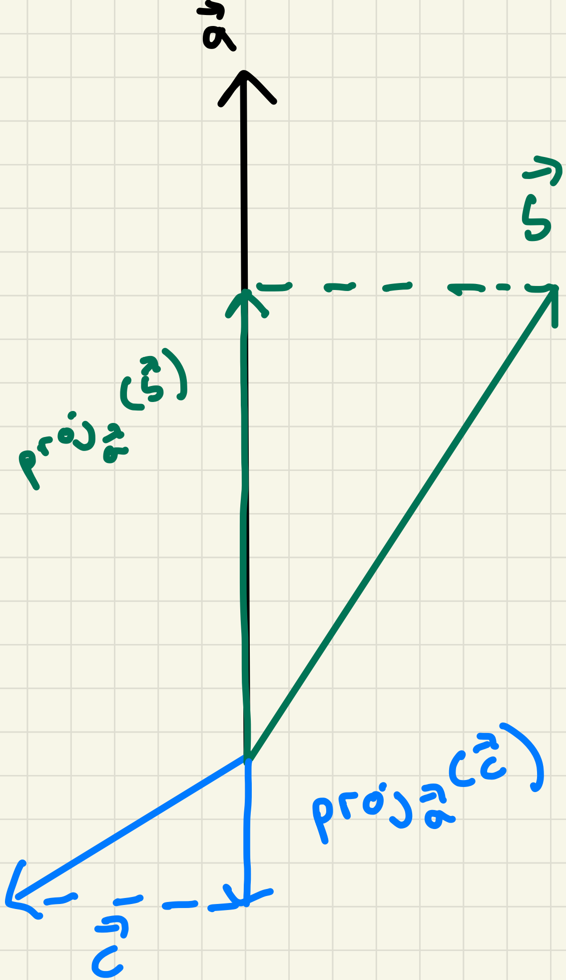


$\text{comp}_{\vec{a}}(\vec{b})$ is a scalar quantity, not a vector

the vector projection is

$$\text{proj}_{\vec{a}}(\vec{b}) = \text{comp}_{\vec{a}}(\vec{b}) \frac{\vec{a}}{|\vec{a}|} \quad \text{(not quite)}$$

$$= \underbrace{\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}}_{\text{scalar}} \underbrace{\frac{\vec{a}}{|\vec{a}|}}_{\text{vector}} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a}$$



again, in general,

$$proj_{\vec{a}}(\vec{b}) \neq proj_{\vec{b}}(\vec{a})$$

Example:

$\vec{a} = \langle 0, 1, 2 \rangle$

$\vec{b} = \langle 3, 5, -7 \rangle$

$$\text{scal}_{\vec{a}}(\vec{b}) = \frac{\vec{a}}{|\vec{a}|} \cdot \vec{b} = \frac{5 - 14}{\sqrt{1 + 4}} = \frac{-9}{\sqrt{5}}$$

$$\text{proj}_{\vec{a}}(\vec{b}) = \text{scal}_{\vec{a}}(\vec{b}) \frac{\vec{a}}{|\vec{a}|} = \frac{-9}{5} \langle 0, 1, 2 \rangle$$

at home: try $\quad \text{scal}_{\vec{b}}(\vec{a}) = \, ?$

$\text{proj}_{\vec{b}}(\vec{a}) = \, ?$