Equations of lines and planes

Reminder: in the plane (or $\mathbb{R}^2$), the equation of a line looks like

$$y = mx + b$$

slope

y-intercept $(0, b)$

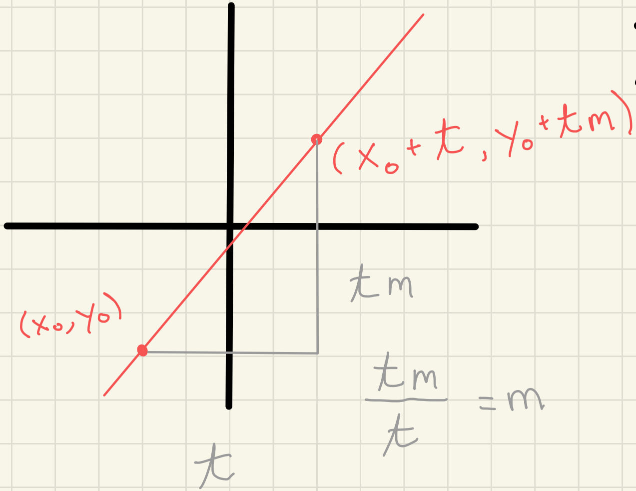
alternatively, we could use point-slope form

$$y - y_0 = m(x - x_0)$$

slope

(x_0, y_0) is a point on the line

$$y - y_0 = m(x - x_0)$$



$$\frac{tm}{t} = m$$

yet another way to describe the line

$$\langle x, y \rangle = \langle x_0, y_0 \rangle + t \langle 1, m \rangle$$

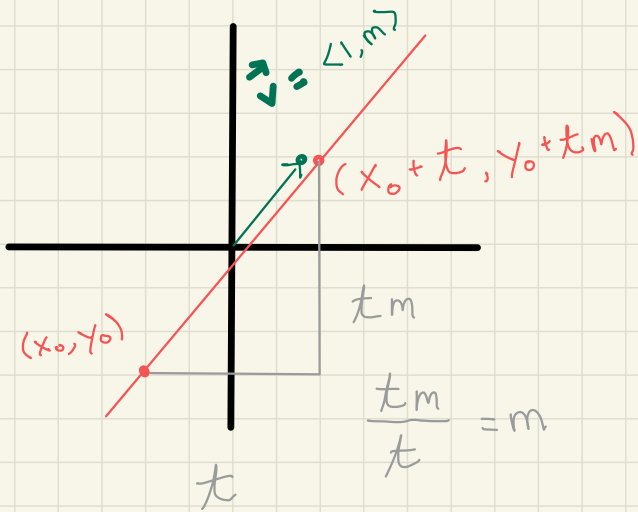
how far to slide along

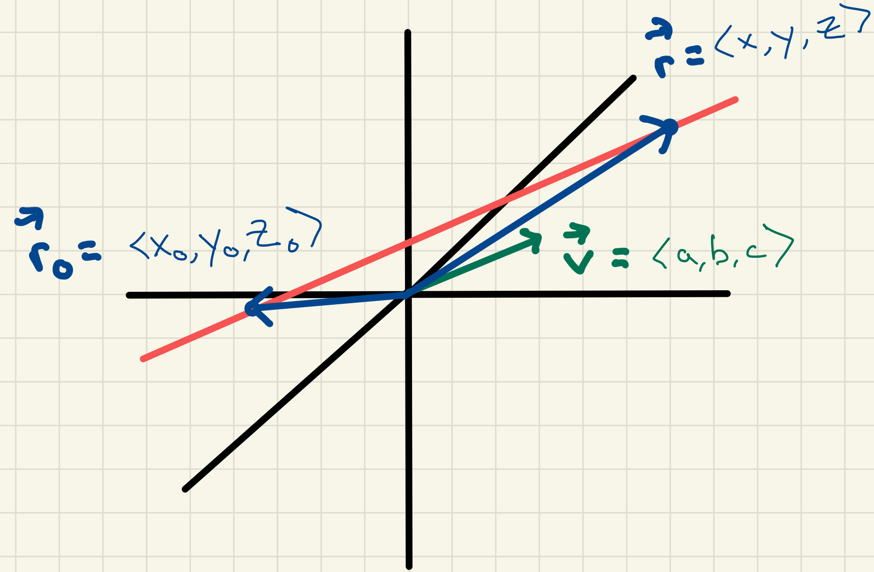
slope

Equation of a line in $\mathbb{R}^2$: $\langle x, y\rangle = \langle x_0, y_0\rangle + t\langle 1, m\rangle$

Equation of a line in $\mathbb{R}^3$: $$\underset{\langle x,y,z\rangle}{\underset{\parallel}{\vec r}} = \underset{\langle x_0,y_0,z_0\rangle}{\underset{\parallel}{\vec r_0}} + t\,\underset{\langle a,b,c\rangle}{\underset{\parallel}{\vec v}}$$

vector eqn of line

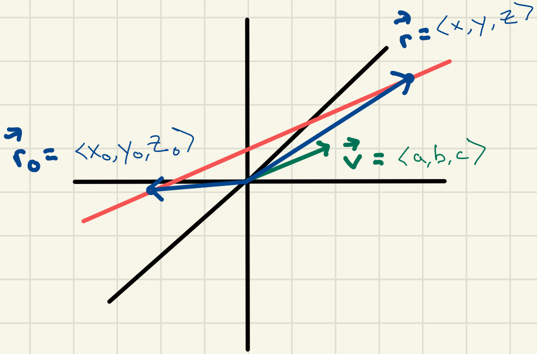




vector eqn of line

$$\vec{r} = \vec{r}_0 + t\vec{v}$$

$$\vec{r} = \langle x, y, z\rangle \qquad \vec{r}_0 = \langle x_0, y_0, z_0\rangle \qquad \vec{v} = \langle a, b, c\rangle$$



$$\langle x, y, z\rangle = \langle x_0 + ta,\ y_0 + tb,\ z_0 + tc\rangle$$

$$x = x_0 + ta$$
$$y = y_0 + tb$$
$$z = z_0 + tc$$

parametric eqns of a line

Note: each value of $t \in \mathbb{R}$ gives a point on the line

Informally, you can think of this as a time parameter

Example: find a vector eqn for the line that passes through the point $(8,6,7)$ and is parallel to the vector $\langle 5,3,9 \rangle$

also find parametric eqns for this line

Ans: $\vec{r} = \vec{r}_0 + t\vec{v}$

$$\vec{r} = \langle 8,6,7 \rangle + t\langle 5,3,9 \rangle = (8+t5)\hat{\imath} + (6+3t)\hat{\jmath} + (7+9t)\hat{k}$$

Ans: $x = 8+5t$, $y = 6+3t$, $z = 7+9t$

Note: many possible answers, can use any vector $\vec{v}$ parallel to $\langle 5,3,9 \rangle$

Example (cont.)

Ans: $\vec{r} = \vec{r}_0 + t\vec{v}$

$$\vec{r} = \langle 8,6,7\rangle + t\langle 5,3,9\rangle = (8+t5)\hat{i} + (6+3t)\hat{j} + (7+9t)\hat{k}$$

Ans: $x = 8+5t$, $y = 6+3t$, $z = 7+9t$

Find two other points on this line

Ans: plug in different values of $t \neq 0$

$t=0 \Rightarrow$ get back original point

$t=1 \Rightarrow \langle 13, 9, 16\rangle$

$t=-1 \Rightarrow \langle 3, 3, -2\rangle$

If a vector $\vec{v} = \langle a, b, c \rangle$ is used to give the direction of a line, then a, b, c are called the direction numbers

Note: not unique, $s\vec{v} = \langle sa, sb, sc \rangle$ also describes the direction of the line for any $s \neq 0$

Ways to write eqn of a line in $\mathbb{R}^3$:

$\vec{r} = \vec{r}_0 + t\vec{v}$ | $x = x_0 + ta, \quad y = y_0 + tb, \quad z = z_0 + tc$

Another way: solve for t $\Rightarrow$ $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$

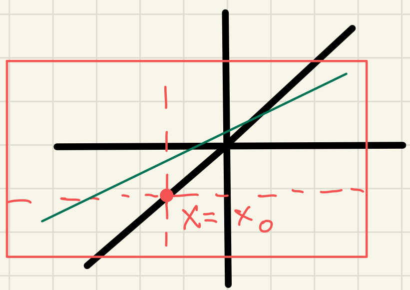
symmetric eqns

Ways to write eqn of a line in $\mathbb{R}^3$:

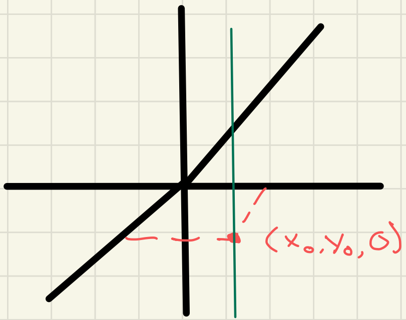
$\vec{r} = \vec{r}_0 + t\vec{v}$ | $x = x_0 + ta, \quad y = y_0 + tb, \quad z = z_0 + tc$

Another way: solve for t $\Rightarrow$ $\dfrac{x-x_0}{a} = \dfrac{y-y_0}{b} = \dfrac{z-z_0}{c}$

If $a = 0$, then $x = x_0$, $\dfrac{y-y_0}{b} = \dfrac{z-z_0}{c}$



If $a = 0$, $b = 0$, then $x = x_0$, $y = y_0$



Example: find parametric eqns and symmetric eqns for the line that passes through $A(0,1,-1)$ and $B(2,5,-3)$

where does this line intersect the xz-plane?

Ans: $\vec{AB} = \langle 2, 4, -2 \rangle = \vec{v}$

$x = 0 + 2t = 2t \quad , \quad y = 1 + 4t \ , \quad z = -1 - 2t$

or $x = 2 + 2t \quad , \quad y = 5 + 4t \ , \quad z = -3 - 2t$

$$\frac{x}{2} = \frac{y-1}{4} = \frac{z+1}{-2}$$

Example (cont.) :

where does this line intersect the xz-plane?

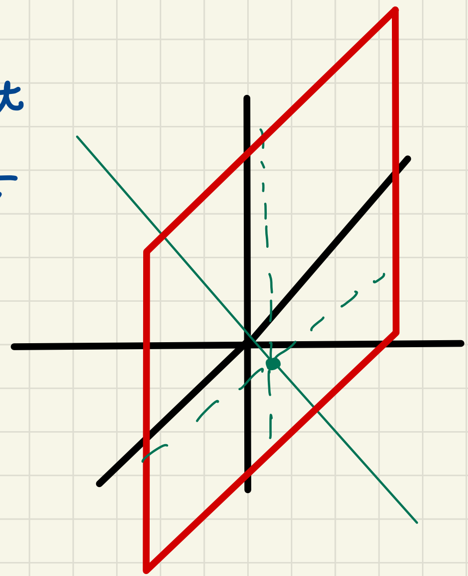
Ans: $\vec{AB} = \langle 2, 4, -2 \rangle = \vec{v}$

(★) $x = 0 + 2t = 2t$, $y = 1 + 4t$, $z = -1 - 2t$

or

$x = 2 + 2t$, $y = 5 + 4t$, $z = -3 - 2t$

$$\frac{x}{2} = \frac{y-1}{4} = \frac{z+1}{-2}$$



xz-plane means $y = 0$

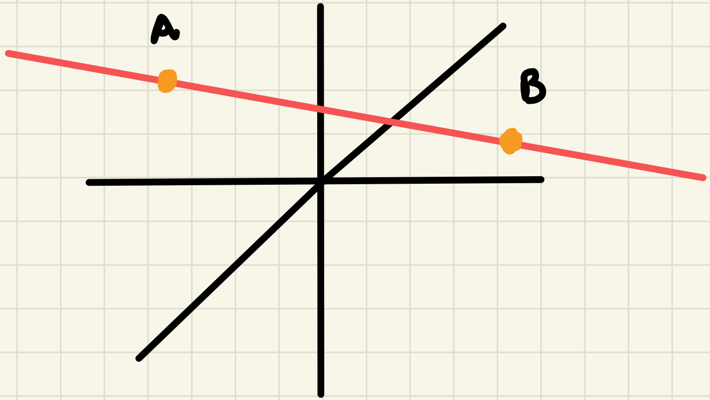
using eqns in (★)

$y = 1 + 4t = 0 \Rightarrow t = \frac{-1}{4} \Rightarrow x = 2t = \frac{-1}{2}$

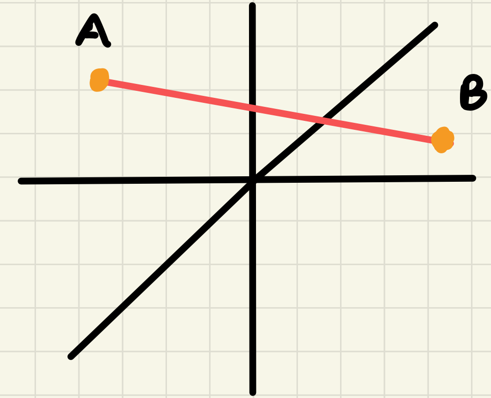
$z = -1 - 2t = -\frac{1}{2}$

@ $\left(-\frac{1}{2}, 0, -\frac{1}{2}\right)$

What if we just want a line segment?



versus



$A(0,1,-1)$, $B(2,5,-3)$

$x = 2t$, $y = 1+4t$, $z = -1-2t$

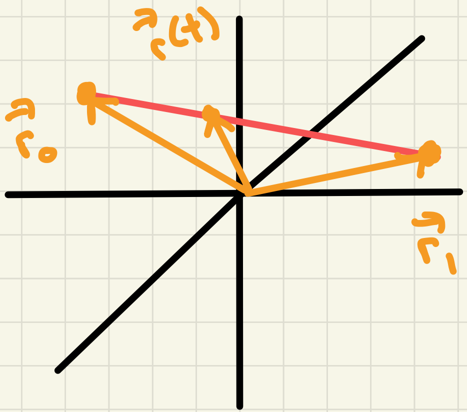
$t=0$? get A

$t=1$? get B

$t \in (0,1)$? somewhere in between

Conclusion: the line segment from $\vec{r}_0$ to $\vec{r}_1$ is given by $\vec{r}(t) = \vec{r}_0 + t(\vec{r}_1 - \vec{r}_0)$

$$= (1-t)\vec{r}_0 + t\vec{r}_1 \quad \text{for } t \in [0,1]$$



Example: show that the lines L_1 and L_2 with parametric eqns

$x = 2 + t$, $y = 3 + 2t$, $z = 1 + 3t$

$x = 2t$, $y = 1 - t$, $z = 2 + t$

are skew lines (i.e., do not intersect and are not parallel)

direction: $\langle 1, 2, 3 \rangle$ direction: $\langle 2, -1, 1 \rangle$

if they intersect,

$$2 + t = 2s \Rightarrow t = 2(s-1)$$
$$3 + 2t = 1 - s \Rightarrow t = \frac{-2-s}{2} = -1 - \frac{s}{2}$$
$$1 + 3t = 2 + s \Rightarrow t = \frac{1+s}{3}$$

Example (cont.)

$t = 2(s-1)$

$t = \frac{-2-s}{2} = -1 - \frac{s}{2}$

$t = \frac{1+s}{3}$

$2(s-1) = -1 - \frac{s}{2}$

$4(s-1) = -2 - s$

$5s = 2$

$s = \frac{2}{5} \Rightarrow t = \frac{-6}{5}$

$-\frac{6}{5} \stackrel{?}{=} \frac{1+\frac{2}{5}}{3}$

no intersection!

Brief summary

Line in $\mathbb{R}^2$: need 2 points

OR

1 point and slope ("direction")

Line in $\mathbb{R}^3$: need 2 points

OR

1 point and vector parallel ("direction")

Intermission

Planes

Line in $\mathbb{R}^3$: need 2 points

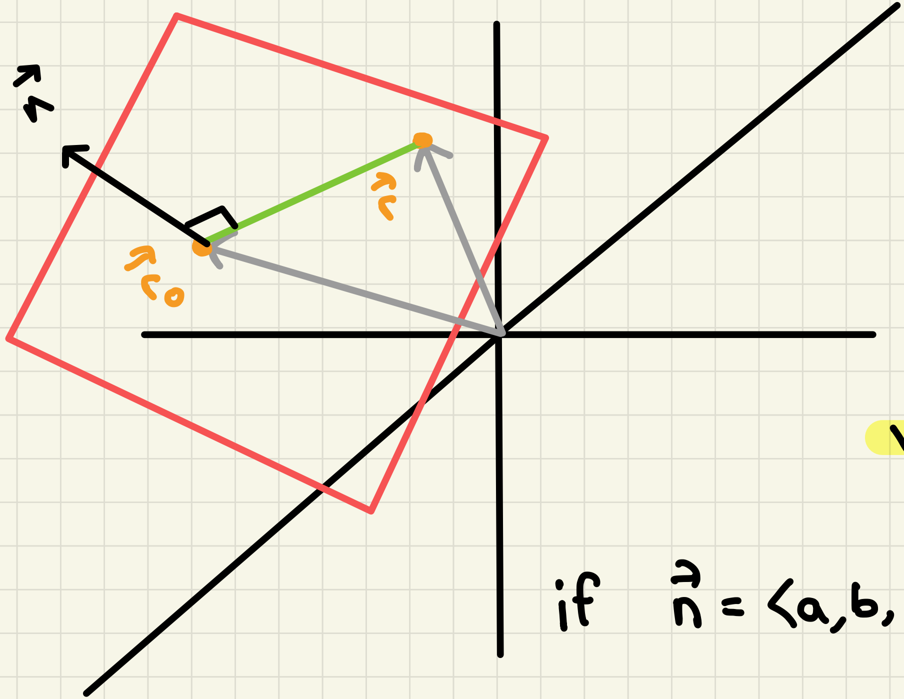
OR

1 point and vector parallel ("direction")

Plane in $\mathbb{R}^3$: need 3 points

OR

1 point and vector PERPENDICULAR



$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

or

$$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$$

vector eqn of a plane

if $\vec{n} = \langle a, b, c \rangle$ and $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

or $$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

scalar eqn of the plane through $P(x_0, y_0, z_0)$
with normal vector $\vec{n} = \langle a, b, c \rangle$

Example: Find an eqn of the plane through the point $P(1,1,1)$ with normal vector $\vec{n} = \langle 2,3,1 \rangle$

Find the intercepts and sketch the plane

Ans: $\langle 2,3,1\rangle \cdot \langle x-1, y-1, z-1 \rangle = 0$

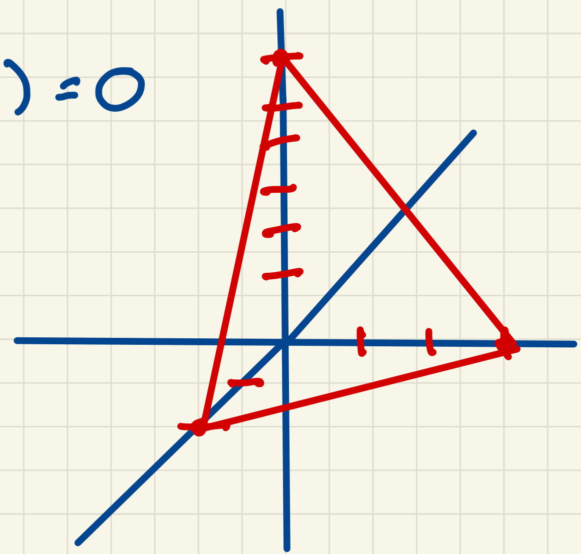
$$2(x-1) + 3(y-1) + (z-1) = 0$$

or $\quad 2x + 3y + z = 6$

x-intercept : $(3,0,0)$

y-intercept : $(0,2,0)$

z-intercept : $(0,0,6)$



Another form of a plane:

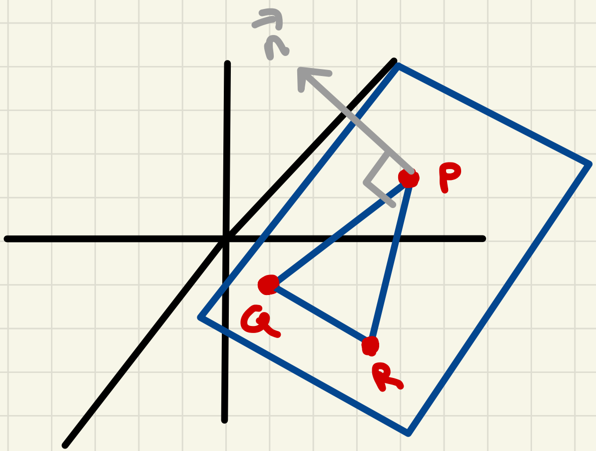
$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

$$ax+by+cz+d=0 \qquad (d=-ax_0-by_0-cz_0)$$

linear eqn in x, y, z

$$ax+by+cz=d \qquad (d=ax_0+by_0+cz_0)$$

Example: find an eqn of the plane that passes through the points $P(0,1,2)$, $Q(3,-1,2)$, and $R(4,-2,1)$



Ans: need a normal vector

take $\vec{PQ} \times \vec{PR} = \langle 3,-2,0 \rangle \times \langle 4,-3,-1 \rangle$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & 0 \\ 4 & -3 & -1 \end{vmatrix} = \langle 2, 3, -1 \rangle$$

$$2(x-0) + 3(y-1) - 1(z-2) = 0$$

Example: Find the point at which the line with parametric eqns $x = 1+4t$, $y = 2+3t$, $z = 3-2t$ intersects the plane $2x+3y+4z = 10$

Ans: plug in eqns for x, y, z

$$2(1+4t) + 3(2+3t) + 4(3-2t) = 10$$

$$20 + 9t = 10$$

$$t = \frac{-10}{9} \Rightarrow \quad x = 1 + 4\left(\frac{-10}{9}\right)$$

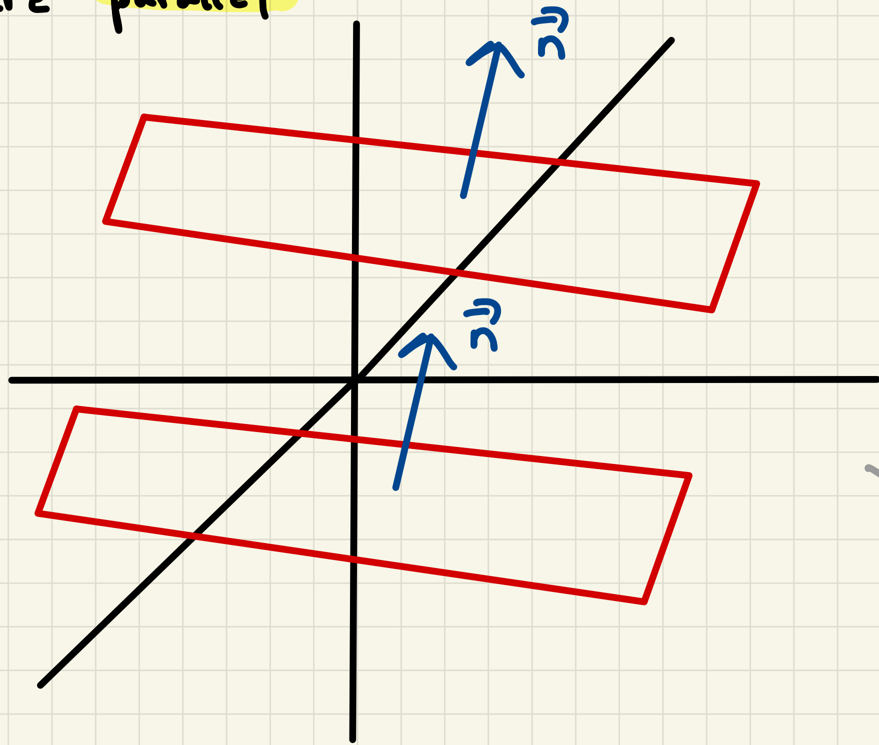
$$y = 2 + 3\left(\frac{-10}{9}\right)$$

$$z = 3 - 2\left(\frac{-10}{9}\right)$$

Intermission

More on planes

two planes are parallel if their normal vectors are parallel



Example:

$$2(x-3)+3(y-1)+(z-2)=0$$

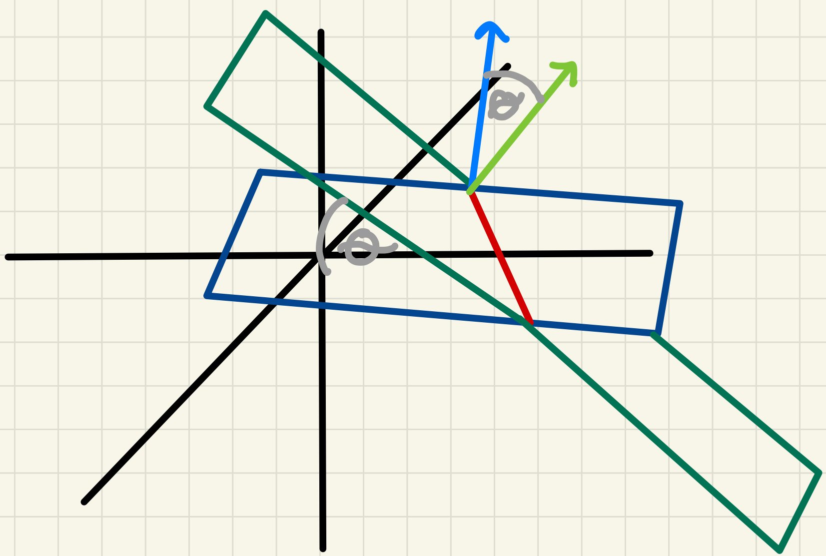
$$10(x-1)+15(y-7)+5(z-7)=0$$

$\langle 2,3,1\rangle$

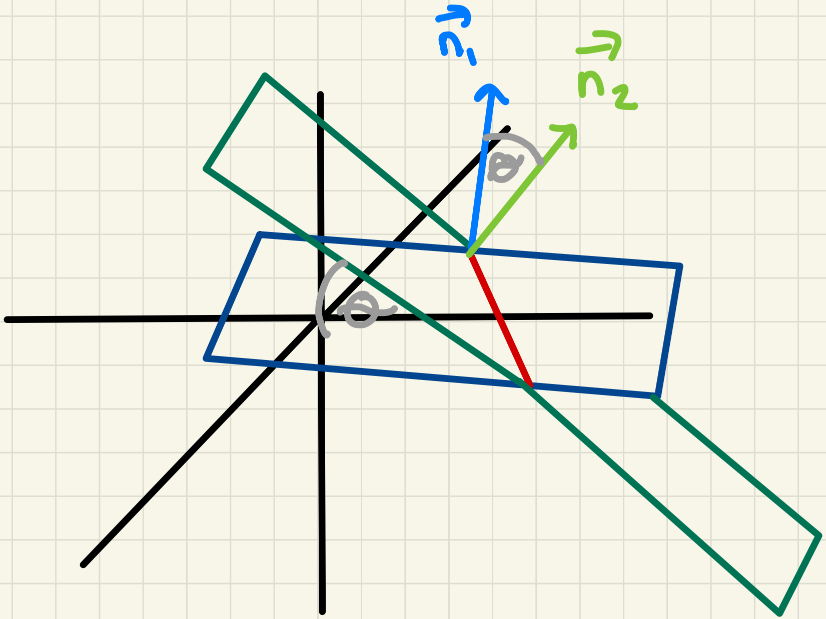
$\langle 10,15,5\rangle$

More on planes

If not parallel, then they intersect in a straight line



the angle between the two planes is defined as the acute angle between their normal vectors how do we actually find this angle?



Ans: dot product!

$$\vec{n}_1 \cdot \vec{n}_2 = |\vec{n}_1||\vec{n}_2| \cos(\theta)$$

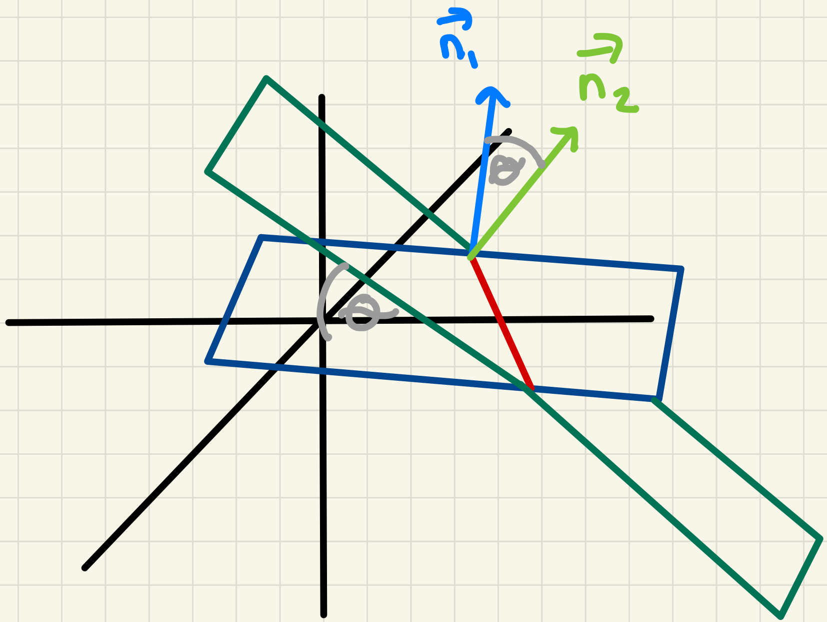
$$\cos(\theta) = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$

$$\theta = \cos^{-1}\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right)$$

Not quite! we want $\theta \in [0, \frac{\pi}{2}]$,

so $\theta = \cos^{-1}\left(\frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}\right)$

how do we find the eqn of the line of intersection?



find a single point

find a vector parallel

this vector must be perpendicular to BOTH $\vec{n}_1$ and $\vec{n}_2$

cross product!

$\vec{n}_1 \times \vec{n}_2$

Example: planes $-2x + y + z = 3$

$x - y - 2z = 4$

angle between?

Ans: $\vec{n}_1 = \langle -2, 1, 1 \rangle$

$\vec{n}_2 = \langle 1, -1, -2 \rangle$

$$\theta = \cos^{-1}\left(\frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1|\,|\vec{n}_2|} \right) = \cos^{-1}\left(\frac{|-2-1-2|}{\sqrt{4+1+1}\,\sqrt{1+1+4}} \right)$$

$$= \cos^{-1}\left(\frac{5}{6} \right)$$

Example: planes $-2x + y + z = 3$

$x - y - 2z = 4$

eqn of line of intersection?

Ans: $\vec{n}_1 = \langle -2, 1, 1 \rangle$, $\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 1 \\ 1 & -1 & -2 \end{vmatrix} = -\hat{i} - 3\hat{j} + \hat{k}$

$\vec{n}_2 = \langle 1, -1, -2 \rangle$

$-2x + y + z = 3$

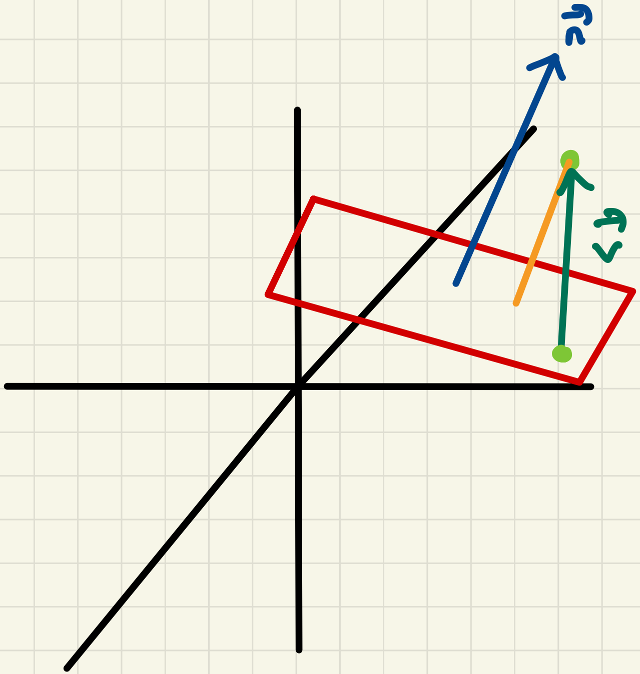
$x - y - 2z = 4$

add together $-x - z = 7$

$z = 0$, $x = -7$, $y = -11$

$x = -7 - t$, $y = -11 - 3t$, $z = t$

Distance between a point and a plane



how to measure

how far $\vec{v}$ travels

in the direction of $\vec{n}$?

absolute value of the

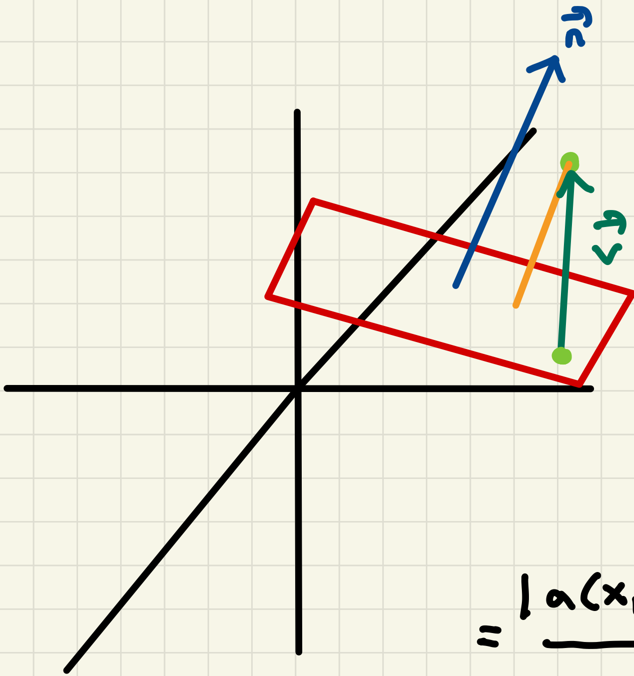
scalar projection!

$$\left| \operatorname{comp}_{\vec{n}}(\vec{v}) \right| = \frac{|\vec{n} \cdot \vec{v}|}{|\vec{n}|}$$

Formula : plane $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

or $ax+by+cz+d=0$

point (x_1, y_1, z_1)



$$\vec{v} = \langle x_1-x_0,\ y_1-y_0,\ z_1-z_0 \rangle$$

$$\vec{n} = \langle a, b, c \rangle$$

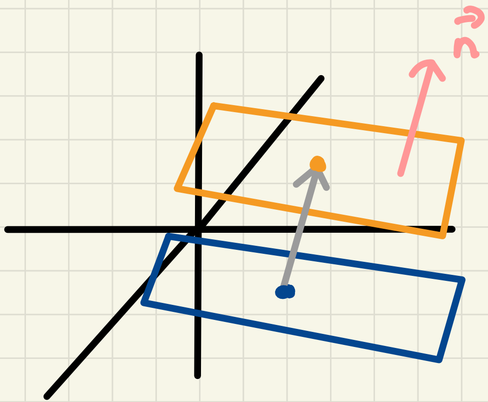
$$|\text{comp}_{\vec{n}}(\vec{v})| = \frac{|\vec{n} \cdot \vec{v}|}{|\vec{n}|}$$

$d = -ax_0 - by_0 - cz_0$

$$= \frac{|a(x_1-x_0)+b(y_1-y_0)+c(z_1-z_0)|}{\sqrt{a^2+b^2+c^2}} = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

Example: find the distance between the parallel planes

$$2x + 3y - z = 1$$
$$4x + 6y - 2z = 3$$



a point on the first plane:

$$x = y = 0, \quad z = -1$$

$$\text{comp}_{\vec{n}}(\vec{v}) = \frac{|4(0) + 6(0) - 2(-1) - 3|}{\sqrt{56}}$$

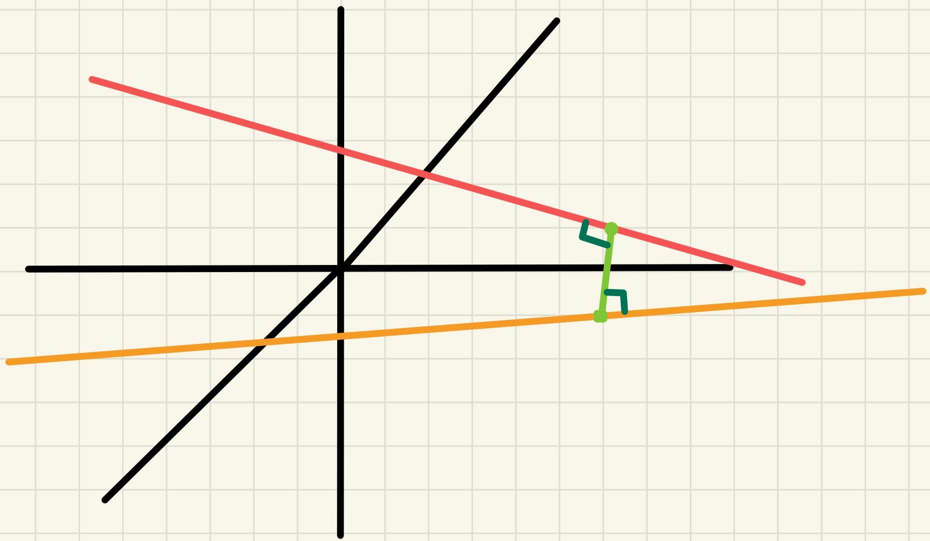
$$= \frac{1}{\sqrt{56}}$$

$\vec{n} = \langle 4, 6, -2 \rangle$

Example: find the distance between the skew lines

$x = 2 + t$, $y = 3 + 2t$, $z = 1 + 3t$

$x = 2t$, $y = 1 - t$, $z = 2 + t$



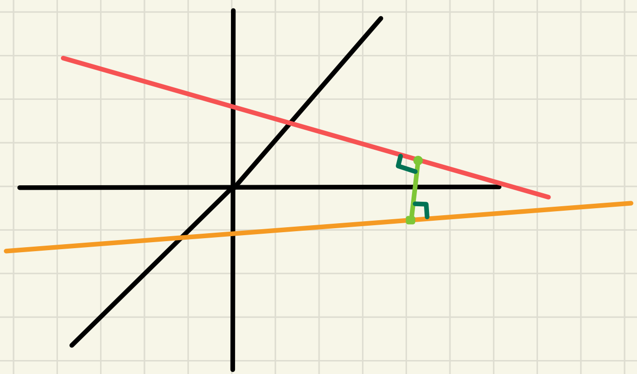
how to get the line line?

cross product!

Example: Find the distance between the skew lines

$x = 2+t \quad , \quad y = 3+2t \quad , \quad z = 1+3t$

$x = 2t \quad , \quad y = 1-t \quad , \quad z = 2+t$



$\langle 1,2,3\rangle \times \langle 2,-1,1\rangle = \langle 5,5,-5\rangle$

point on first line: $(2,3,1)$

point on second line: $(0,1,2)$

plane that has the second line with normal vector $\langle 5,5,-5\rangle$:

$5(x-0)+5(y-1)-5(z-2)=0$

or $5x+5y-5z+5=0$, $d=5$

distance: $\dfrac{|5(2)+5(3)-5(1)+5|}{\sqrt{5^2+5^2+(-5)^2}}$