

Equations of lines and planes

Reminder : in the plane (or \mathbb{R}^2), the equation of a line looks like

$$y = mx + b$$

slope \downarrow m \leftarrow b y-intercept

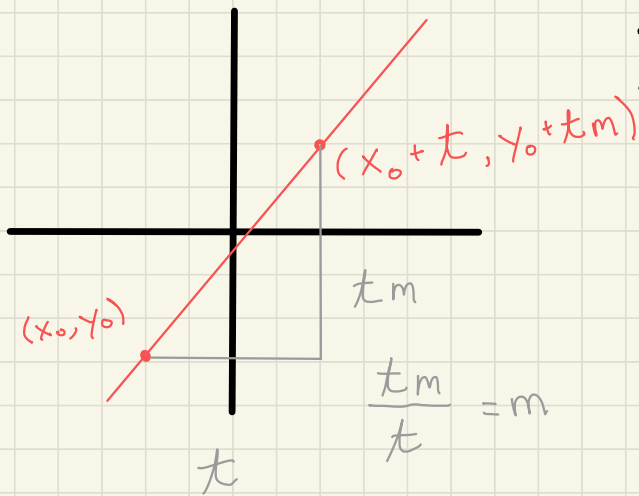
alternatively, we could use point-slope form

$$y - y_0 = m(x - x_0)$$

slope \uparrow m

(x_0, y_0) is a point on the line

$$y - y_0 = m(x - x_0)$$



yet another way to describe the line

$$(x, y) = (x_0, y_0) + t(1, m)$$

how far
to slide along

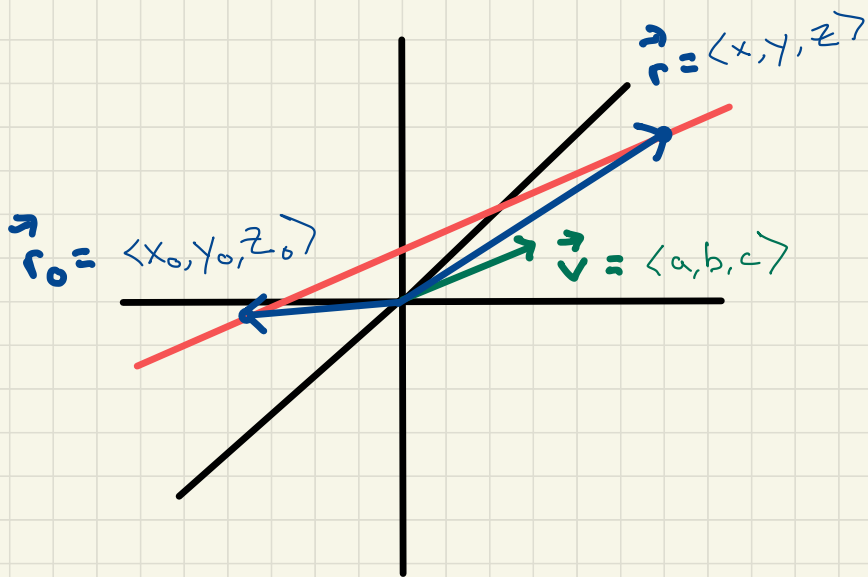
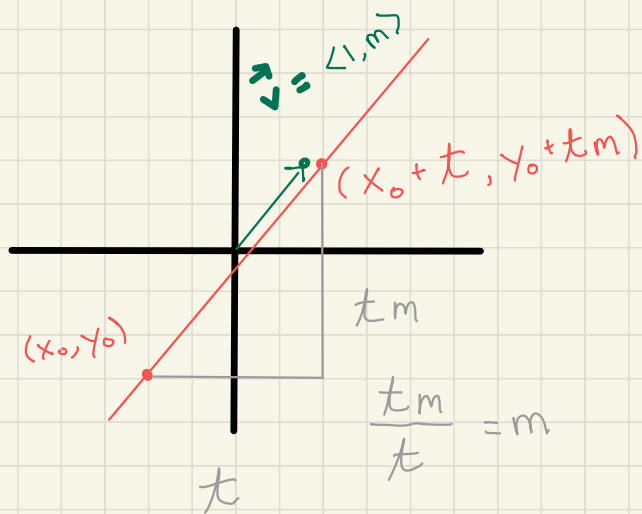
slope

Equation of a line in \mathbb{R}^2 : $(x, y) = (x_0, y_0) + t(1, m)$

Equation of a line in \mathbb{R}^3 : $\vec{r} = \vec{r}_0 + t\vec{v}$

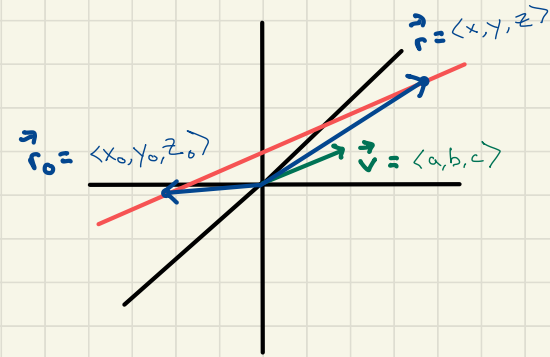
vector eqn of line

$\vec{r} = \langle x, y, z \rangle$ $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$ $\vec{v} = \langle a, b, c \rangle$



vector eqn of line

$$\begin{array}{l} \vec{r} = \vec{r}_0 + t\vec{v} \\ \text{"} \quad \quad \quad \text{"} \quad \quad \quad \text{"} \\ \langle x, y, z \rangle \quad \langle x_0, y_0, z_0 \rangle \quad \langle a, b, c \rangle \end{array}$$



$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

$$x = x_0 + ta$$

$$y = y_0 + tb$$

$$z = z_0 + tc$$

parametric eqns of
a line

Note: each value of $t \in \mathbb{R}$ gives a point on the line
Informally, you can think of this as a time parameter

Example: find a vector eqn for the line that passes through the point $(8, 6, 7)$ and is parallel to the vector $\langle 5, 3, 9 \rangle$
also find parametric eqns for this line

Ans: $\vec{r} = \vec{r}_0 + t\vec{v}$

$$\vec{r} = \langle 8, 6, 7 \rangle + t \langle 5, 3, 9 \rangle = (8+5t)\hat{i} + (6+3t)\hat{j} + (7+9t)\hat{k}$$

Ans: $x = 8 + 5t$, $y = 6 + 3t$, $z = 7 + 9t$

Note: many possible answers, can use any vector \vec{v} parallel to $\langle 5, 3, 9 \rangle$

Example (cont.)

Ans: $\vec{r} = \vec{r}_0 + t\vec{v}$

$$\vec{r} = \langle 8, 6, 7 \rangle + t \langle 5, 3, 9 \rangle = (8+5t)\hat{i} + (6+3t)\hat{j} + (7+9t)\hat{k}$$

Ans: $x = 8+5t$, $y = 6+3t$, $z = 7+9t$

Find two other points on this line

Ans: plug in different values of $t \neq 0$

$t = 0 \Rightarrow$ get back original point

$t = 1 \Rightarrow \langle 13, 9, 16 \rangle$

$t = -1 \Rightarrow \langle 3, 3, -2 \rangle$

If a vector $\vec{v} = \langle a, b, c \rangle$ is used to give the direction of a line, then a, b, c are called the **direction numbers**

Note: not unique, $s\vec{v} = \langle sa, sb, sc \rangle$ also describes the direction of the line for any $s \neq 0$

Ways to write eqn of a line in \mathbb{R}^3 :

$$\vec{r} = \vec{r}_0 + t\vec{v} \quad | \quad x = x_0 + ta, \quad y = y_0 + tb, \quad z = z_0 + tc$$

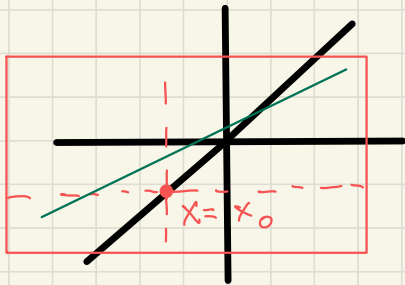
Another way: solve for $t \Rightarrow \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$
Symmetric eqns

Ways to write eqn of a line in \mathbb{R}^3 :

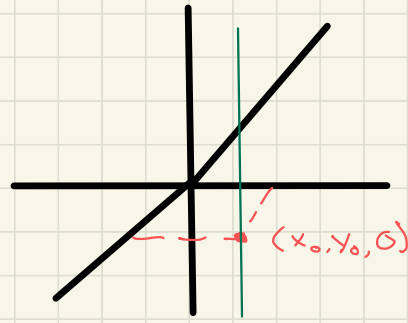
$$\vec{r} = \vec{r}_0 + t\vec{v} \quad | \quad x = x_0 + ta, \quad y = y_0 + tb, \quad z = z_0 + tc$$

Another way: solve for $t \Rightarrow \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$

If $a=0$, then $x=x_0$, $\frac{y-y_0}{b} = \frac{z-z_0}{c}$



If $a=0, b=0$, then $x=x_0, y=y_0$



Example: find parametric eqns and symmetric eqns for the line that passes through $A(0, 1, -1)$ and $B(2, 5, -3)$

where does this line intersect the xz -plane?

Ans: $\vec{AB} = \langle 2, 4, -2 \rangle = \vec{v}$

$$x = 0 + 2t = 2t, \quad y = 1 + 4t, \quad z = -1 - 2t$$

or

$$x = 2 + 2t, \quad y = 5 + 4t, \quad z = -3 - 2t$$

$$\frac{x}{2} = \frac{y-1}{4} = \frac{z+1}{-2}$$

Example (cont.):

where does this line intersect the xz -plane?

Ans: $\vec{AB} = \langle 2, 4, -2 \rangle = \vec{v}$

(★) $x = 0 + 2t = 2t$, $y = 1 + 4t$, $z = -1 - 2t$

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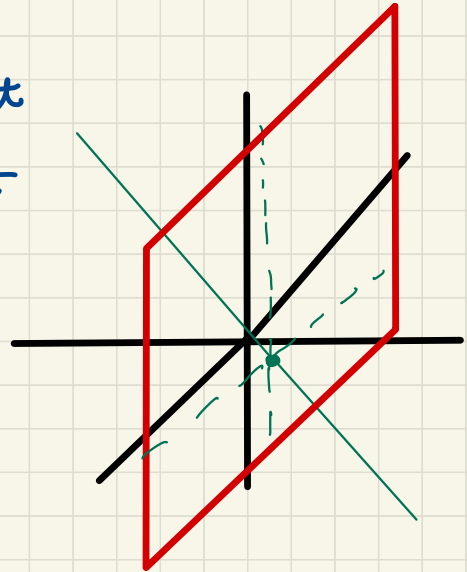
xz -plane means $y = 0$

using eqns in (★)

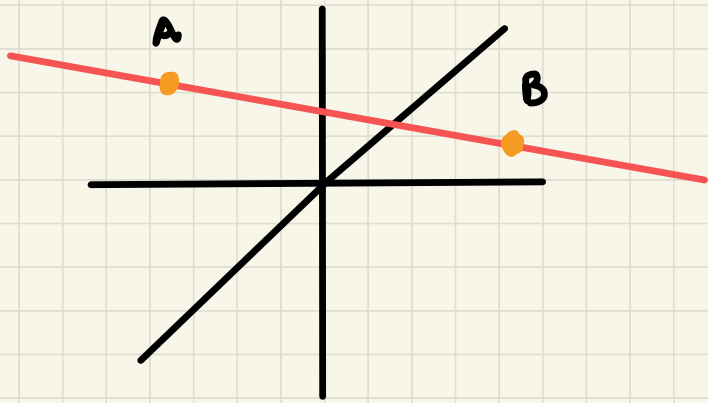
$$y = 1 + 4t = 0 \Rightarrow t = -\frac{1}{4} \Rightarrow x = 2t = -\frac{1}{2}$$

$$z = -1 - 2t = -\frac{1}{2}$$

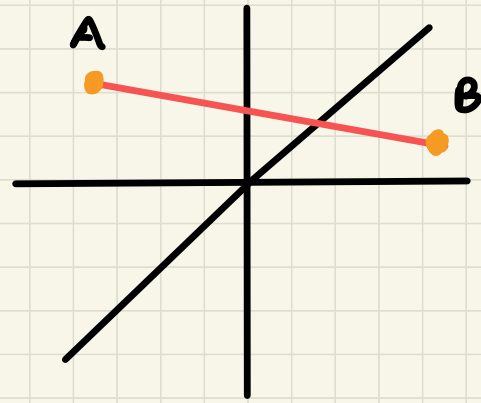
@ $(-\frac{1}{2}, 0, -\frac{1}{2})$



What if we just want a line segment?



versus



$$A(0, 1, -1), \quad B(2, 5, -3)$$

$$x = 2t, \quad y = 1 + 4t, \quad z = -1 - 2t$$

$t = 0$? get A

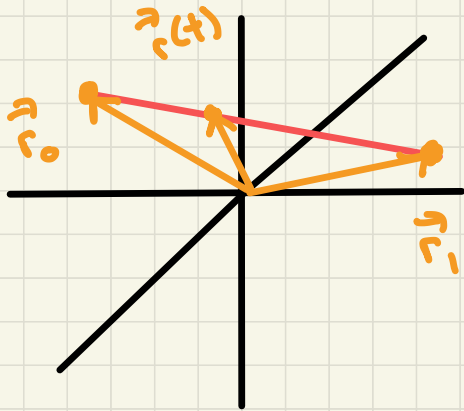
$t = 1$? get B

$t \in (0, 1)$? somewhere in between

Conclusion: the line segment from \vec{r}_0 to \vec{r}_1 ,

is given by $\vec{r}(t) = \vec{r}_0 + t(\vec{r}_1 - \vec{r}_0)$

$$= (1-t)\vec{r}_0 + t\vec{r}_1 \quad \text{for } t \in [0, 1]$$



Example: show that the lines L_1 and L_2 with parametric eqns

$$x = 2 + t, \quad y = 3 + 2t, \quad z = 1 + 3t$$

$$x = 2t, \quad y = 1 - t, \quad z = 2 + t$$

are skew lines (i.e., do not intersect and are not parallel)

direction: $\langle 1, 2, 3 \rangle$

direction: $\langle 2, -1, 1 \rangle$

if they intersect,

$$\begin{aligned} 2 + t &= 2s & \Rightarrow & t = 2(s - 1) \\ 3 + 2t &= 1 - s & \Rightarrow & t = \frac{-2 - s}{2} = -1 - \frac{s}{2} \\ 1 + 3t &= 2 + s & \Rightarrow & t = \frac{1 + s}{3} \end{aligned}$$

Example (cont.)

$$x = 2(s-1)$$

$$x = \frac{-2-s}{2} = -1 - \frac{s}{2}$$

$$x = \frac{1+s}{3}$$



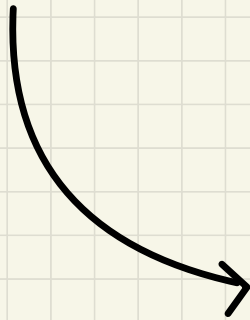
$$2(s-1) = -1 - \frac{s}{2}$$

$$4(s-1) = -2 - s$$

$$5s = 2$$

$$s = \frac{2}{5}$$

$$\Rightarrow x = \frac{-6}{5}$$



$$-\frac{6}{5} \stackrel{?}{=} \frac{1 + \frac{2}{5}}{3}$$

no intersection!

Brief summary

Line in \mathbb{R}^2 : need 2 points

OR

1 point and slope ("direction")

Line in \mathbb{R}^3 : need 2 points

OR

1 point and vector parallel ("direction")

Intermission

Planes

Line in \mathbb{R}^3 : need 2 points

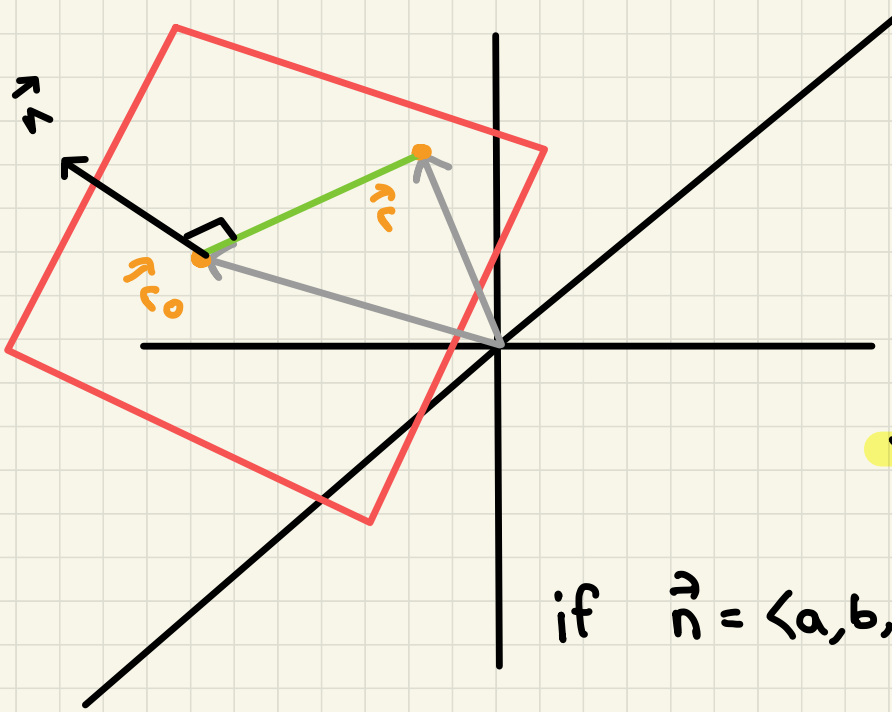
OR

1 point and vector parallel ("direction")

Plane in \mathbb{R}^3 : need 3 points

OR

1 point and vector PERPENDICULAR



$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

or

$$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$$

vector eqn of a plane

$$\text{if } \vec{n} = \langle a, b, c \rangle \text{ and } \vec{r}_0 = \langle x_0, y_0, z_0 \rangle$$

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

or

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

scalar eqn of the plane through $P(x_0, y_0, z_0)$
with normal vector $\vec{n} = \langle a, b, c \rangle$

Example : find an eqn of the plane through
the point $P(1,1,1)$ with normal vector $\vec{n} = \langle 2, 3, 1 \rangle$

find the intercepts and sketch the plane

Ans: $\langle 2, 3, 1 \rangle \cdot \langle x-1, y-1, z-1 \rangle = 0$

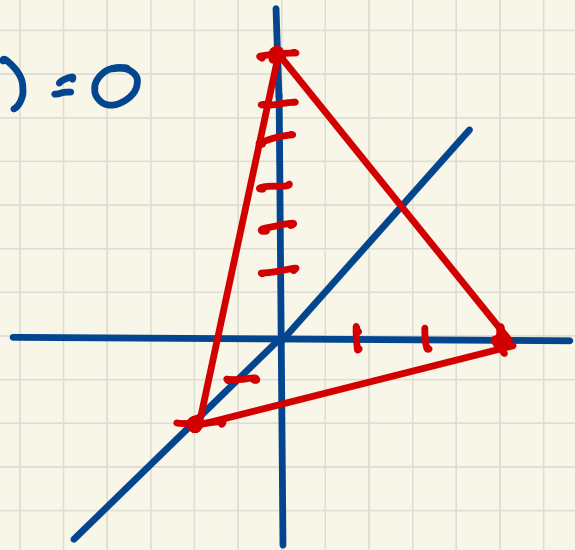
$$2(x-1) + 3(y-1) + (z-1) = 0$$

or $2x + 3y + z = 6$

x-intercept : $(3, 0, 0)$

y-intercept : $(0, 2, 0)$

z-intercept : $(0, 0, 6)$



Another form of a plane :

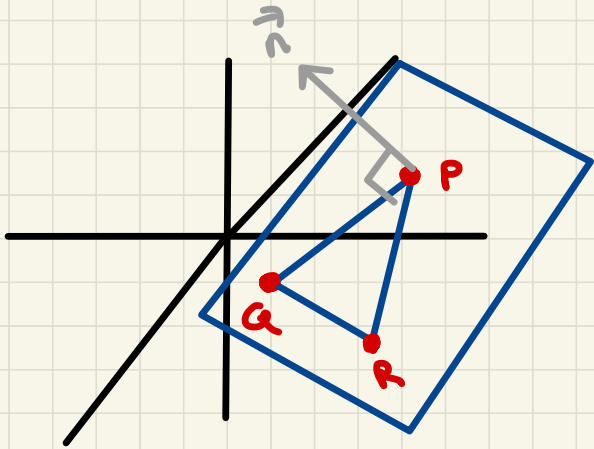
$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$ax + by + cz + d = 0 \quad (d = -ax_0 - by_0 - cz_0)$$

linear eqn in x, y, z

$$ax + by + cz = d \quad (d = ax_0 + by_0 + cz_0)$$

Example: find an eqn of the plane that passes through the points $P(0,1,2)$, $Q(3,-1,2)$, and $R(4,-2,1)$



Ans: need a normal vector

take $\vec{PQ} \times \vec{PR} = \langle 3, -2, 0 \rangle \times \langle 4, -3, -1 \rangle$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & 0 \\ 4 & -3 & -1 \end{vmatrix} = \langle 2, 3, -1 \rangle$$

$$2(x-0) + 3(y-1) - 1(z-2) = 0$$

Example: find the point at which the line
with parametric eqns $x = 1 + 4t$, $y = 2 + 3t$, $z = 3 - 2t$
intersects the plane $2x + 3y + 4z = 10$

Ans: plug in eqns for x, y, z

$$2(1+4t) + 3(2+3t) + 4(3-2t) = 10$$

$$26 + 9t = 10$$

$$t = \frac{-10}{9} \Rightarrow$$

$$x = 1 + 4\left(\frac{-10}{9}\right)$$

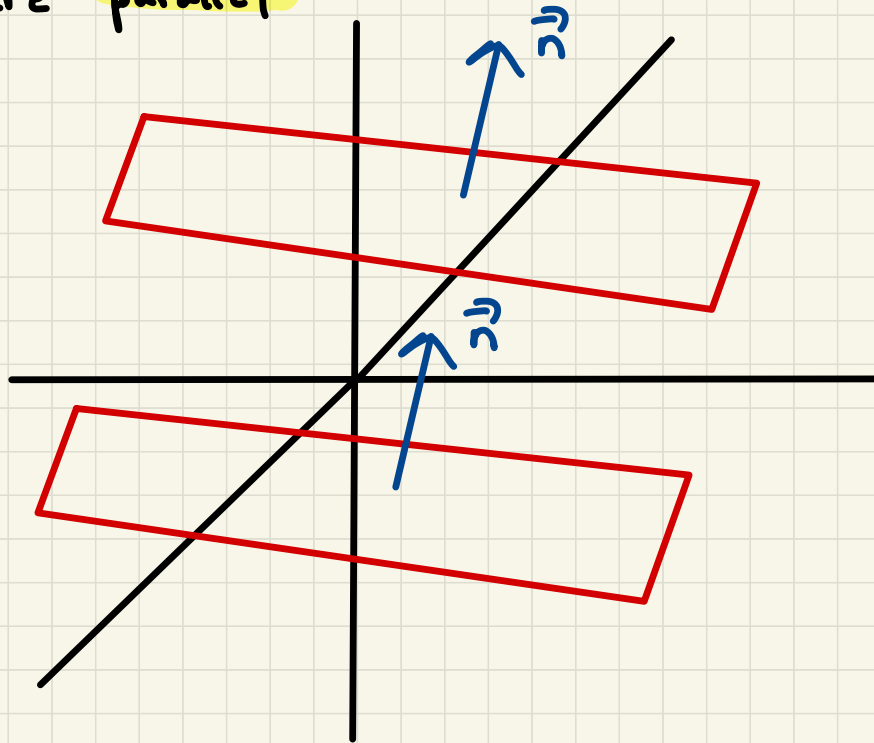
$$y = 2 + 3\left(\frac{-10}{9}\right)$$

$$z = 3 - 2\left(\frac{-10}{9}\right)$$

Intermission

More on planes

two planes are **parallel** if their normal vectors are **parallel**



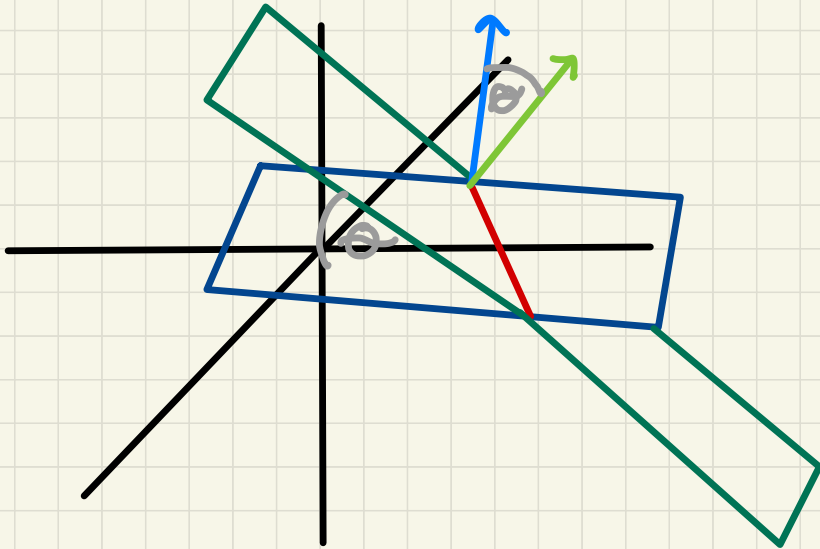
Example:

$$2(x-3)+3(y-1)+(z-2)=0$$

$$10(x-1)+15(y-7)+5(z-7)=0$$

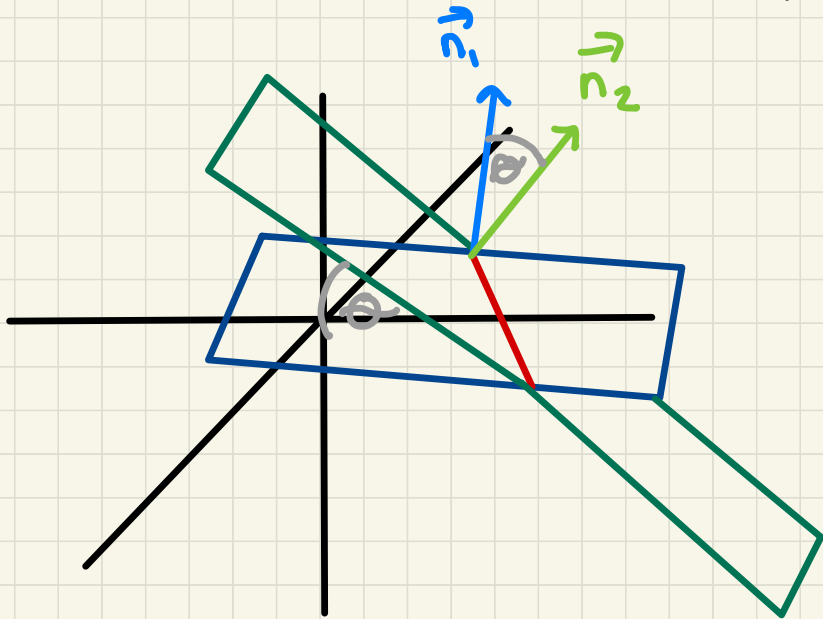
More on planes

If not parallel, then they intersect in a straight line



the angle between the two planes is defined as the acute angle between their normal vectors

how do we actually find this angle?



Ans: dot product!

$$\vec{n}_1 \cdot \vec{n}_2 = |\vec{n}_1| |\vec{n}_2| \cos(\theta)$$

$$\cos(\theta) = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|}$$

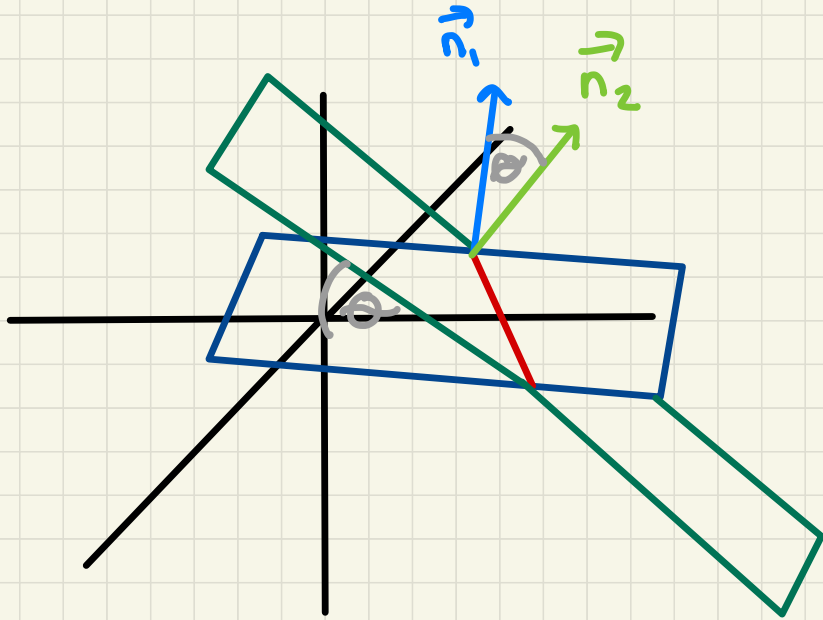
$$\theta = \cos^{-1} \left(\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|} \right)$$

Not quite!

we want $\theta \in [0, \frac{\pi}{2}]$,

so $\theta = \cos^{-1} \left(\frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1| |\vec{n}_2|} \right)$

how do we find the eqn of the
line of intersection?



find a single point
find a vector parallel
this vector must be
perpendicular to BOTH

\vec{n}_1 and \vec{n}_2

Cross product!

$$\vec{n}_1 \times \vec{n}_2$$

Example: planes $-2x + y + z = 3$
 $x - y - 2z = 4$

angle between?

Ans: $\vec{n}_1 = \langle -2, 1, 1 \rangle$

$\vec{n}_2 = \langle 1, -1, -2 \rangle$

$$\begin{aligned} \Phi &= \cos^{-1} \left(\frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1| |\vec{n}_2|} \right) = \cos^{-1} \left(\frac{|-2-1-2|}{\sqrt{4+1+1} \sqrt{1+1+4}} \right) \\ &= \cos^{-1} \left(\frac{5}{6} \right) \end{aligned}$$

Example: planes $-2x + y + z = 3$
 $x - y - 2z = 4$

eqn of line of intersection?

Ans: $\vec{n}_1 = \langle -2, 1, 1 \rangle$, $\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 1 \\ 1 & -1 & -2 \end{vmatrix} = -\hat{i} - 3\hat{j} + \hat{k}$
 $\vec{n}_2 = \langle 1, -1, -2 \rangle$

$$\begin{aligned} -2x + y + z &= 3 \\ x - y - 2z &= 4 \end{aligned}$$

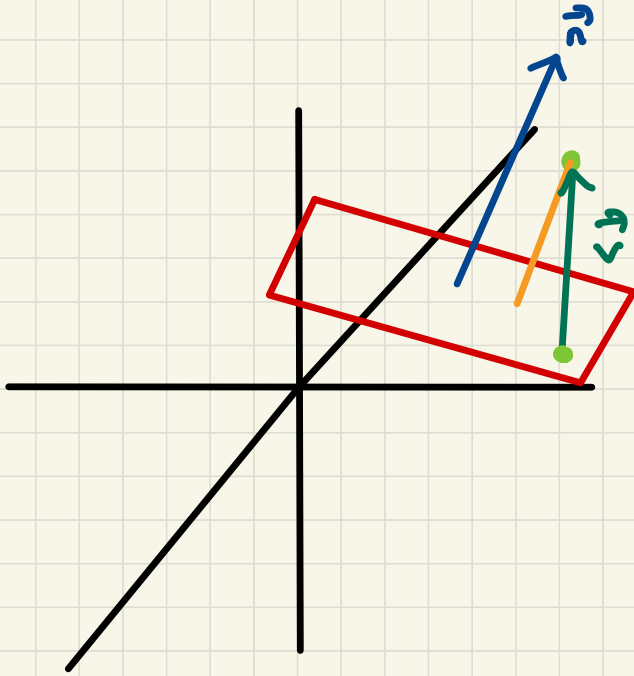
add together

$$-x - z = 7$$

$$z = 0, x = -7, y = -11$$

$$x = -7 - t, y = -11 - 3t, z = t$$

Distance between a point and a plane



how to measure

how far \vec{v} travels

in the direction of \vec{n} ?

absolute value of the
scalar projection!

$$|\text{comp}_{\vec{n}}(\vec{v})| = \frac{|\vec{n} \cdot \vec{v}|}{|\vec{n}|}$$

Formula : plane $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$
point (x_1, y_1, z_1)

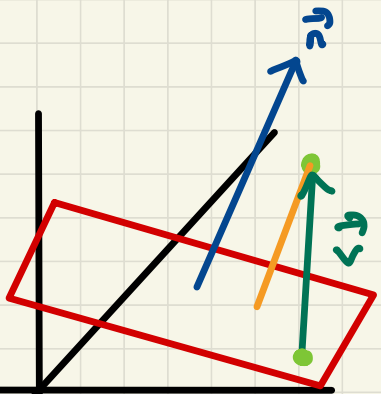
$$\vec{v} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$$

$$\vec{n} = \langle a, b, c \rangle$$

$$|\text{comp}_{\vec{n}}(\vec{v})| = \frac{|\vec{n} \cdot \vec{v}|}{|\vec{n}|}$$

$$d = \begin{matrix} -ax_0 \\ -by_0 \\ -cz_0 \end{matrix}$$

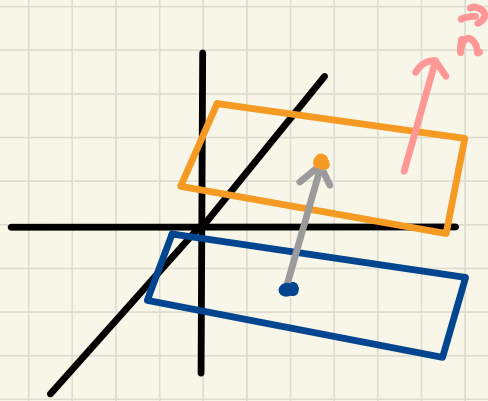
$$= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$



Example: find the distance between the parallel planes

$$2x + 3y - z = 1$$

$$4x + 6y - 2z = 3$$



a point on the first plane:

$$x = y = 0, z = -1$$

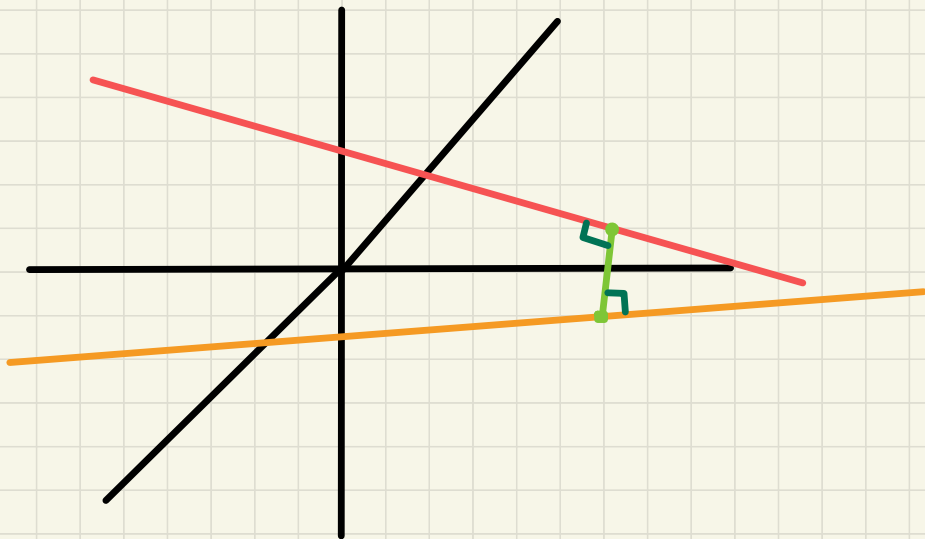
$$\vec{n} = \langle 4, 6, -2 \rangle$$

$$\begin{aligned} \text{comp}_{\vec{n}}(\vec{v}) &= \frac{|4(0) + 6(0) - 2(-1) - 3|}{\sqrt{56}} \\ &= \frac{1}{\sqrt{56}} \end{aligned}$$

Example: find the distance between the skew lines

$$x = 2 + t, \quad y = 3 + 2t, \quad z = 1 + 3t$$

$$x = 2t, \quad y = 1 - t, \quad z = 2 + t$$



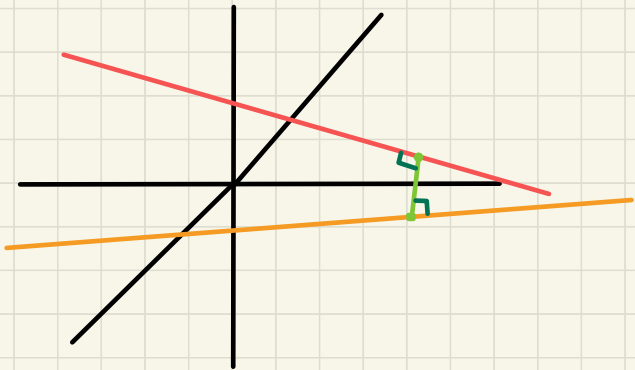
how to get
the line
line?

cross product!

Example: find the distance between the skew lines

$$x = 2 + t, \quad y = 3 + 2t, \quad z = 1 + 3t$$

$$x = 2t, \quad y = 1 - t, \quad z = 2 + t$$



$$\langle 1, 2, 3 \rangle \times \langle 2, -1, 1 \rangle = \langle 5, 5, -5 \rangle$$

point on first line: $(2, 3, 1)$

point on second line: $(0, 1, 2)$

plane that has the second line with normal vector $\langle 5, 5, -5 \rangle$:

$$5(x-0) + 5(y-1) - 5(z-2) = 0$$

$$\text{or } 5x + 5y - 5z + 5 = 0, \quad d = 5$$

$$\text{distance: } \frac{|5(2) + 5(3) - 5(1) + 5|}{\sqrt{5^2 + 5^2 + (-5)^2}}$$