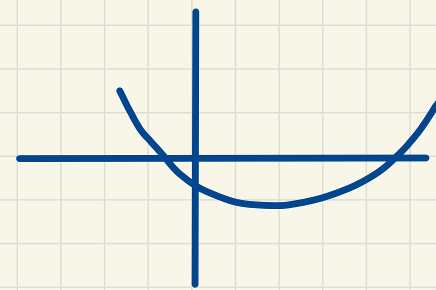
Maximum and minimum values

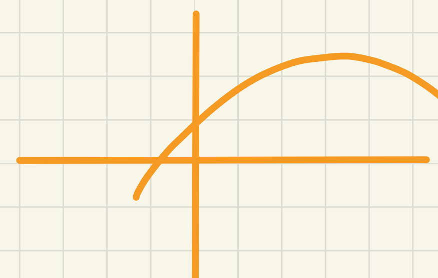
Maximum and minimum values for functions of a single variable: recall the second derivative test

if $f'(c) = 0$

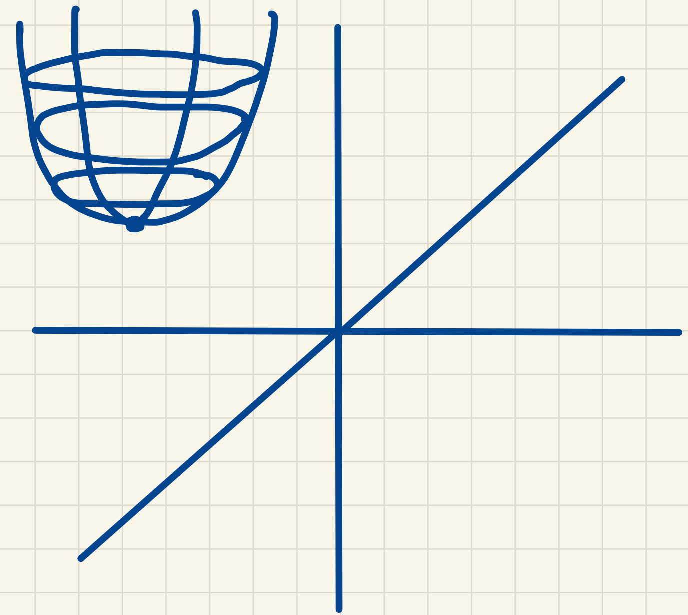
and $f''(c) > 0$ then f has a local min at $x = c$



and $f''(c) < 0$ then f has a local max at $x = c$



Maximum and minimum values for functions of two variables:



$f(x,y)$ has a local min at (a,b) if $f(x,y) \geq f(a,b)$ when (x,y) is near (a,b)

the value $f(a,b)$ is called a local min value (if $f(x,y) \geq f(a,b)$ for all (x,y) in domain, then absolute min)

Maximum and minimum values for functions of two variables:

$f(x,y)$ has a local max at (a,b) if $f(x,y) \le f(a,b)$ when (x,y) is near (a,b)

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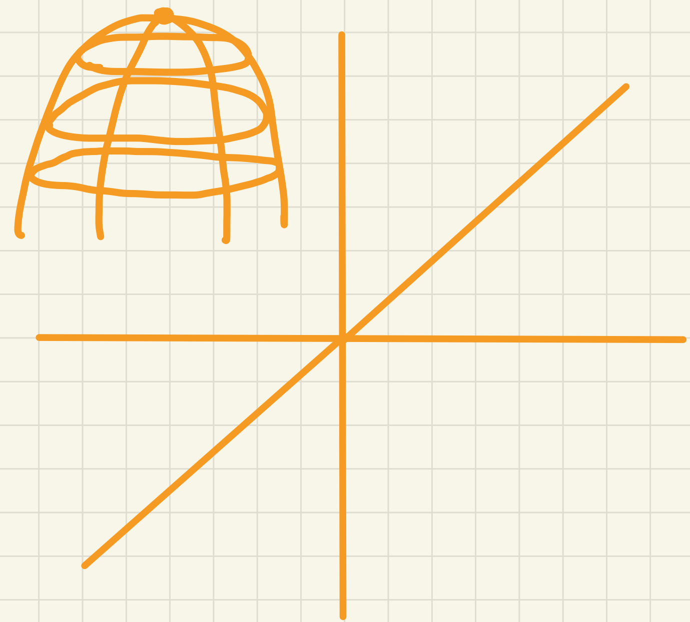

Theorem (Fermat): if f has a local min or local max at (a,b) and the first-order partials exist at (a,b), then

$$f_x(a,b) = f_y(a,b) = 0$$

A point (a,b) is called a critical point (or stationary point) if $f_x(a,b) = f_y(a,b) = 0$ or if one of these partial derivatives does not exist

If f has a local min or max at (a,b), then (a,b) is a critical point of f

The converse is not true, even in the single-variable case (recall $f(x)=x^3$)

A critical point could be a local min, a local max, or neither

How to tell?

Example: let $f(x,y) = x^2 + y^2 - 4x - 8y + 3$

find the critical points and classify them

Ans:

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find the critical points and classify them

Ans:

$$f_x(x,y) = 2x - 4 = 0 \iff x = 2$$

$$f_y(x,y) = 2y - 8 = 0 \iff y = 4$$

$$f(x,y) = x^2 + y^2 - 4x - 8y + 3 = \underbrace{(x-2)^2}_{\geq 0} + \underbrace{(y-4)^2}_{\geq 0} - 17$$

so $f(x,y) \geq -17$ and $f(2,4) = -17$ so $(2,4)$ is an absolute min

Example: find and classify the critical points of $f(x,y) = y^2 - x^2$

Ans:

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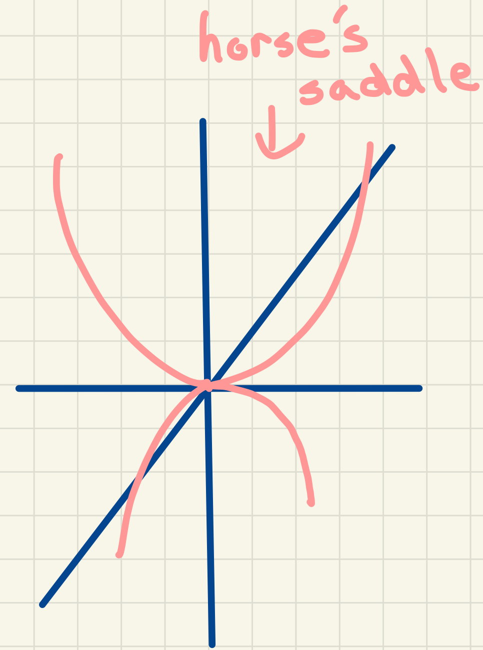
Ans: $f_x(x,y) = -2x = 0 \iff x = 0$

$f_y(x,y) = 2y = 0 \iff y = 0$

$f(0,0) = 0$, not a min

not a max



Second derivatives test: suppose the second partial derivatives of f are continuous on a disk with center (a,b) and suppose that

$$f_x(a,b) = f_y(a,b) = 0$$

let $D = D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2$

- if $D > 0$ and $f_{xx}(a,b) > 0$, then local min
- if $D > 0$ and $f_{xx}(a,b) < 0$, then local max
- if $D < 0$, then $f(a,b)$ is neither (saddle point)

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$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$, where $f_{xy} = f_{yx}$ by Clairaut's theorem

if $D = 0$, then no info

Example: find the local max values, local min values, and saddle points of $f(x,y) = x^4 + y^4 - 4xy + 1$

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Example: find the local max values, local min values, and saddle points of $f(x,y) = x^4 + y^4 - 4xy + 1$

Ans: $\begin{cases} f_x = 4x^3 - 4y = 0 \iff x^3 = y \\ f_y = 4y^3 - 4x = 0 \iff x = y^3 \end{cases}$ combine

$$y^9 = y$$

$$y^9 - y = 0$$

$$y(y^8 - 1) = 0$$

$$y = 0 \quad \text{or} \quad y^8 = 1$$

$$\downarrow$$

$$y = \pm 1$$

critical points: $(0,0)$, $(1,1)$, $(-1,-1)$

Example: find the local max values, local min values, and saddle points of $f(x,y) = x^4 + y^4 - 4xy + 1$

Ans: $f_x = 4x^3 - 4y = 0$

critical points: $(0,0)$, $(1,1)$, $(-1,-1)$

$f_y = 4y^3 - 4x = 0$

$f_{xx} = 12x^2$, $f_{yy} = 12y^2$, $f_{xy} = -4$

$D(0,0) = f_{xx}(0,0)\, f_{yy}(0,0) - f_{xy}(0,0)^2 = -16$ saddle

$D(1,1) = \underset{>0}{f_{xx}(1,1)}\, f_{yy}(1,1) - f_{xy}(1,1)^2 = 12^2 - 16 > 0$ local min

$f(1,1) = -1$

$D(-1,-1) = \underset{>0}{f_{xx}(-1,-1)}\, f_{yy}(-1,-1) - f_{xy}(-1,-1)^2 = 12^2 - 16 > 0$ local min

$f(-1,-1) = -1$

Example: find the shortest distance from the point $(1,0,-2)$ to the plane $x+2y+z=4$

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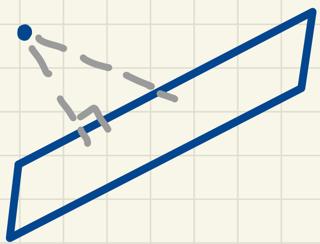

distance from $(1,0,-2)$ to a point (x,y,z) is

$$\sqrt{(x-1)^2+(y-0)^2+(z+2)^2}$$

minimize $(x-1)^2+(y-0)^2+(z+2)^2$ instead

$z=4-x-2y$

minimize $(x-1)^2+y^2+(6-x-2y)^2$

Example: find the shortest distance from the point $(1,0,-2)$ to the plane $x+2y+z=4$

Ans: minimize $$(x-1)^2 + y^2 + (6-x-2y)^2$$ $$\overset{\shortparallel}{f(x,y)}$$

$$f_x(x,y) = 2(x-1) + 2(6-x-2y)(-1) = 0 \iff x-1 = 6-x-2y$$
$$\iff 2x = 7-2y$$

$$f_y(x,y) = 2y + 2(6-x-2y)(-2) = 0 \iff y = 2(6-x-2y)$$
$$\iff 2x = 12-5y$$

$$7-2y = 12-5y \iff 3y = 5 \iff y = \frac{5}{3} \implies x = \frac{11}{6}$$

Example: find the shortest distance from the point $(1,0,-2)$ to the plane $x+2y+z=4$

Ans: minimize $\underbrace{(x-1)^2+y^2+(6-x-2y)^2}_{f(x,y)}$

$y=\frac{5}{3}$, $x=\frac{11}{6}$

$f_x(x,y)=2(x-1)+2(6-x-2y)(-1)$

$f_y(x,y)=2y+2(6-x-2y)(-2)$

$f_{xx}=2+2=4>0$

$f_{yy}=2+8=10$

$f_{xy}=4$

$4\cdot 10-4^2>0$ always

so, $\left(\frac{11}{6},\frac{5}{3}\right)$ is a local min (in fact, absolute min)

$\sqrt{f\left(\frac{11}{6},\frac{5}{3}\right)}=\sqrt{\left(\frac{5}{6}\right)^2+\left(\frac{5}{3}\right)^2+\left(\frac{5}{6}\right)^2}=\frac{5}{6}\sqrt{6}$

Example: A rectangular box WITHOUT a lid is to be made from $12\ m^2$ of cardboard find the max volume of such a box

Ans:

Example: A rectangular box WITHOUT a lid is to be made from 12 m^2 of cardboard find the max volume of such a box

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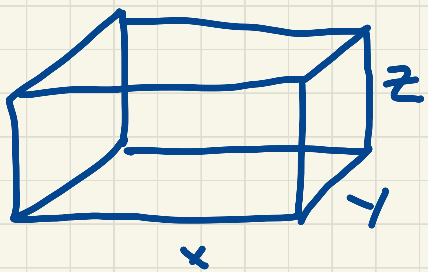


$$2yz + 2xz + xy = 12$$

maximize xyz

$$z(2y + 2x) = 12 - xy$$

$$z = \frac{12 - xy}{2(x+y)}$$

$$f(x,y) = xy\,\frac{12 - xy}{2(x+y)}$$

Example: A rectangular box WITHOUT a lid is to be made from 12 m^2 of cardboard find the max volume of such a box

Ans:

$$f(x,y) = xy\,\frac{12-xy}{2(x+y)} = \frac{12xy - x^2y^2}{2(x+y)}$$

$$f_x = \frac{2(x+y)(12y-2xy^2) - (12xy - x^2y^2)(2)}{(2(x+y))^2}$$

$$= \frac{-x^2y^2 + 12y^2 - 2xy^3}{2(x+y)^2}$$

$$= \frac{y^2(-2xy - x^2 + 12)}{2(x+y)^2}$$

Example: A rectangular box WITHOUT a lid is to be made from 12 m^2 of cardboard find the max volume of such a box

Ans: $f(x,y) = xy\,\frac{12-xy}{2(x+y)} = \frac{12xy - x^2y^2}{2(x+y)}$

$f_x = \frac{y^2(-2xy - x^2 + 12)}{2(x+y)^2}$ $\qquad$ $f_y = \frac{x^2(-2xy - y^2 + 12)}{2(x+y)^2}$

$y^2(-2xy - x^2 + 12) = 0 \Rightarrow -2xy - x^2 + 12 = 0$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\; \| \quad \Rightarrow x^2 = y^2 \Rightarrow x = y$

$x^2(-2xy - y^2 + 12) = 0 \Rightarrow -2xy - y^2 + 12 = 0$

$\Downarrow$

$-3x^2 + 12 = 0$

$\Downarrow$

$x = y = 2$

Example: A rectangular box WITHOUT a lid is to be made from 12 m^2 of cardboard find the max volume of such a box

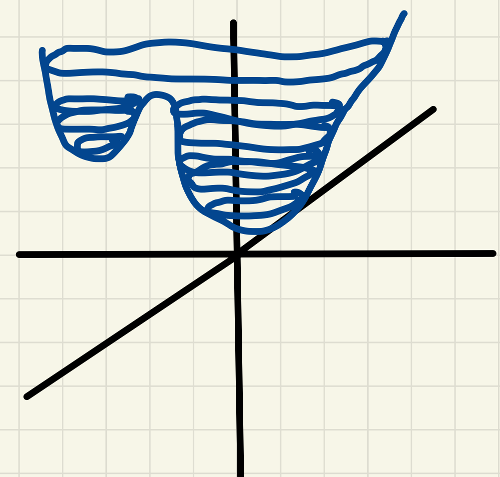
Ans: $x = y = 2$, $z = \frac{12 - xy}{2(x+y)} = 1$

max volume is $4\ m^3$

no need to do second derivative test since we can argue that a maximum has to exist given the physical nature of the problem

Second derivatives test gives info about local max/min

What about absolute max/min?



Recall: for a continuous function of a single variable on a closed interval $[a, b]$, the Extreme Value Theorem says there is both an absolute max and absolute min

Procedure: find critical points and evaluate the function at the critical points and the boundary (i.e., $f(a)$ and $f(b)$)

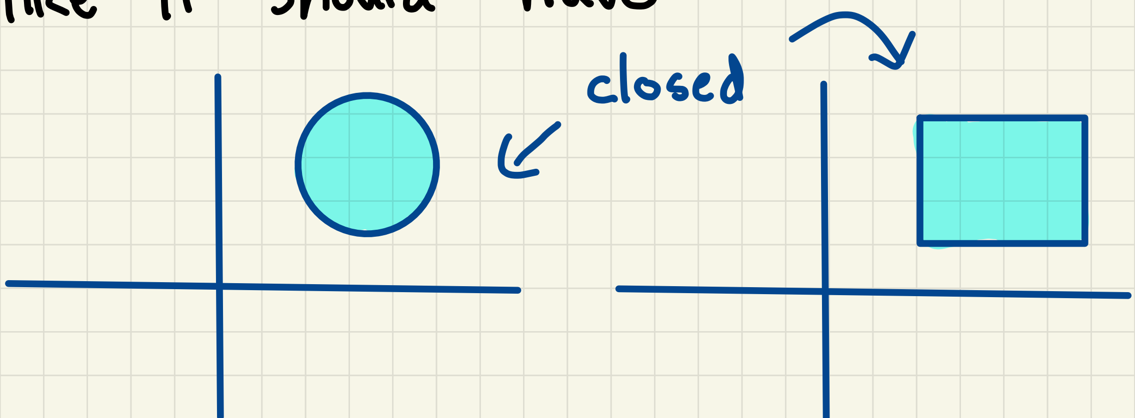
largest is abs max, smallest is abs min

What about two variables?

A closed set in $\mathbb{R}^2$ is a set that contains all of its boundary points

Informally, the set isn't missing anything it looks like it should have

Examples:



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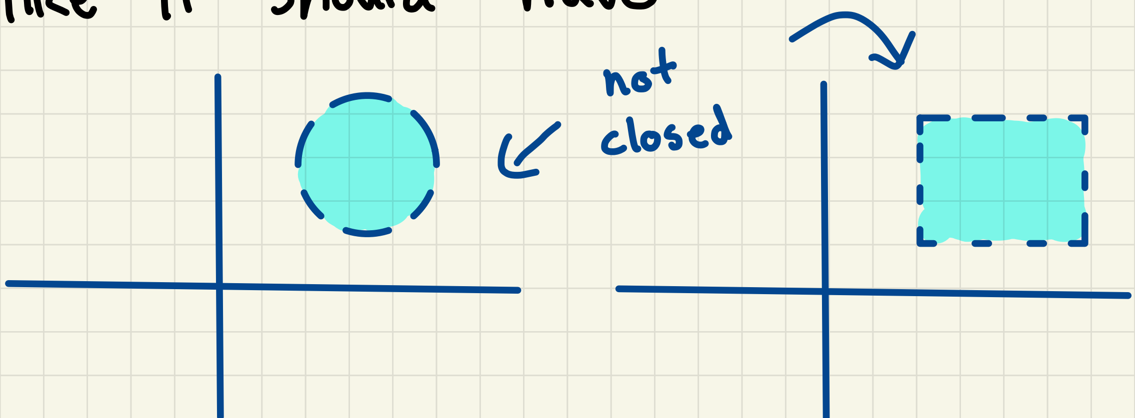
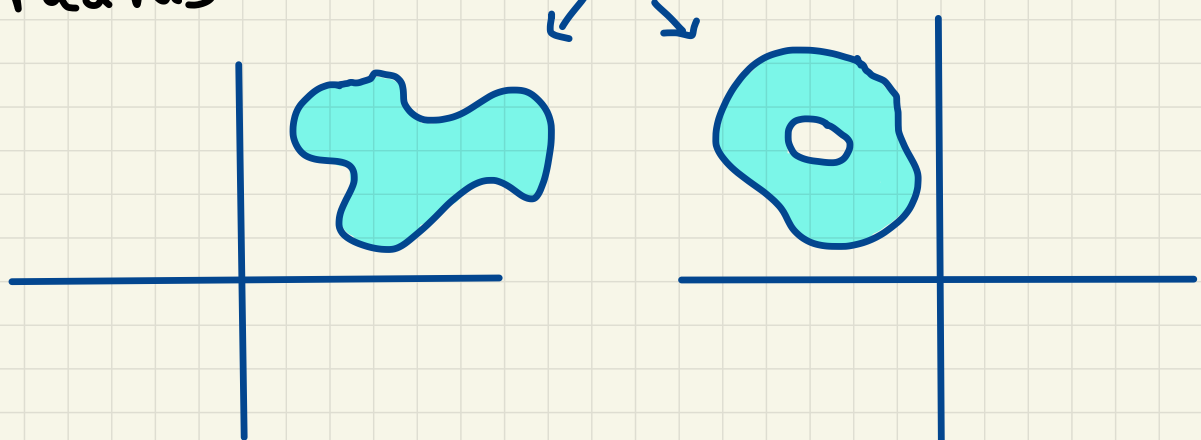


A bounded set in $\mathbb{R}^2$ is one that is contained in some disk

Informally, you can trap it in a disk of finite radius

Examples:

bounded

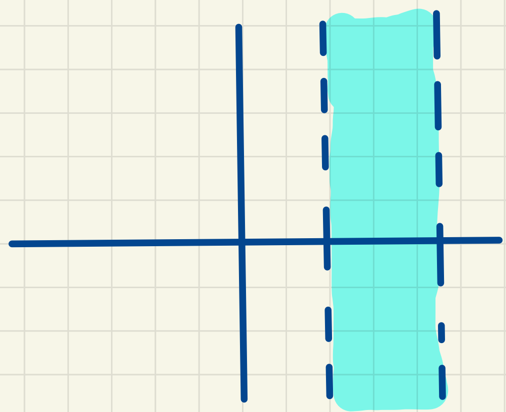


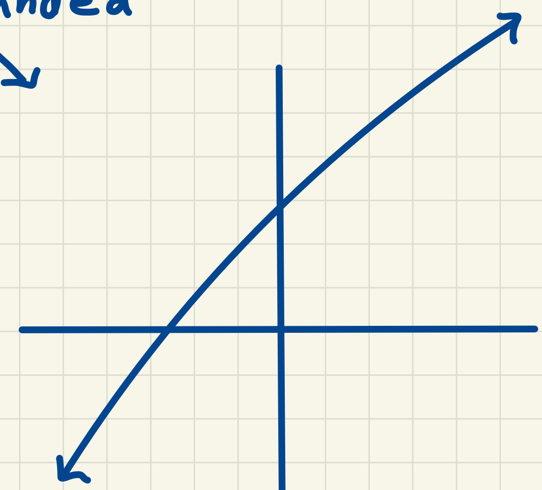
A bounded set in $\mathbb{R}^2$ is one that is contained in some disk

Informally, you can trap it in a disk of finite radius

Examples:

not bounded





Extreme value theorem for functions of two variables:

if f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an abs max $f(x_1,y_1)$ and abs min $f(x_2,y_2)$ at some points $(x_1,y_1),(x_2,y_2) \in D$

Procedure: find the values of f at the critical points in D

find the extreme values of f on the boundary of D

the largest of these values is the abs max

the smallest of these values is the abs min

Example: find the abs max and abs min of

$f(x,y) = x^2 - 2xy + 2y$ on the triangle

$D = \{ (x,y) : 0 \leq x \leq 1, \ 0 \leq y \leq 2x \}$

Ans:

Example: find the abs max and abs min of

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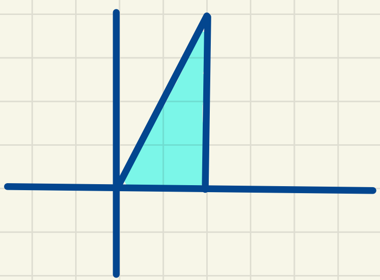

boundary is

$y=0, \quad 0 \le x \le 1$

$x=1, \quad 0 \le y \le 2$

$y=2x, \quad 0 \le x \le 1$

$f_x = 2x - 2y = 0 \iff x=y$

$f_y = -2x + 2 = 0 \iff x = 1 \Rightarrow y = 1$

$(1,1)$ is the only critical pt

Example: find the abs max and abs min of

$f(x,y) = x^2 - 2xy + 2y$ on the triangle

$D = \{(x,y) : 0 \le x \le 1, \; 0 \le y \le 2x\}$

Ans: $(1,1)$ is the only critical pt

$f(1,1) = 1$

boundary

$y=0, \quad 0 \le x \le 1$

$x=1, \quad 0 \le y \le 2$

$y=2x, \quad 0 \le x \le 1$

$f(x,2x) = -3x^2 + 4x$

$f_x(x,2x) = -6x + 4 = 0 \iff x = \frac{2}{3}$

$f(x,0) = x^2, \quad 0 \le x \le 1$

max $f(1,0) = 1$

min $f(0,0) = 0$

$f(1,y) = 1$

max is 1

min is 1

abs max is $f\left(\frac{2}{3}, \frac{4}{3}\right) = \frac{4}{3}$

abs min is $f(0,0) = 0$

Example: find the abs max and abs min of

$f(x,y) = x^2 - 2xy + 2y$ on the triangle

$D = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 2x\}$

Ans: $(1,1)$ is the only critical pt

$f(1,1) = 1$

abs max is $\frac{4}{3}$

abs min is 0

$f(x,2x) = -3x^2 + 4x$

$f_x(x,2x) = -6x + 4 = 0 \iff x = \frac{2}{3}$

max is $f\left(\frac{2}{3}, \frac{4}{3}\right) = \frac{4}{3}$

min is $f(0,0) = 0$

$f(x,0) = x^2, \quad 0 \le x \le 1$

max $f(1,0) = 1$

min $f(0,0) = 0$

$f(1,y) = 1$

max is 1

min is 1