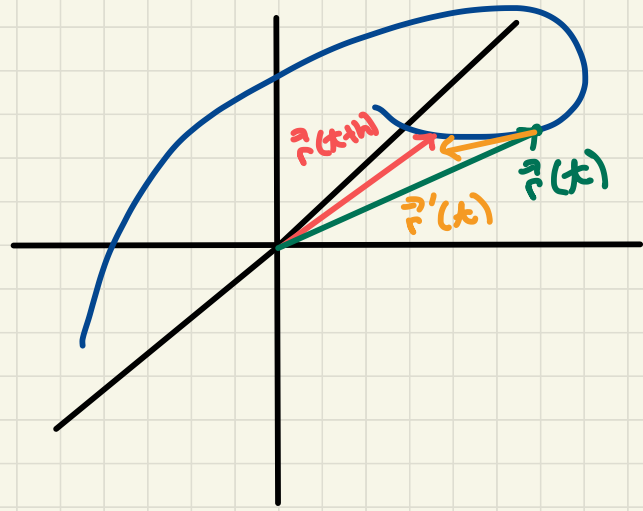
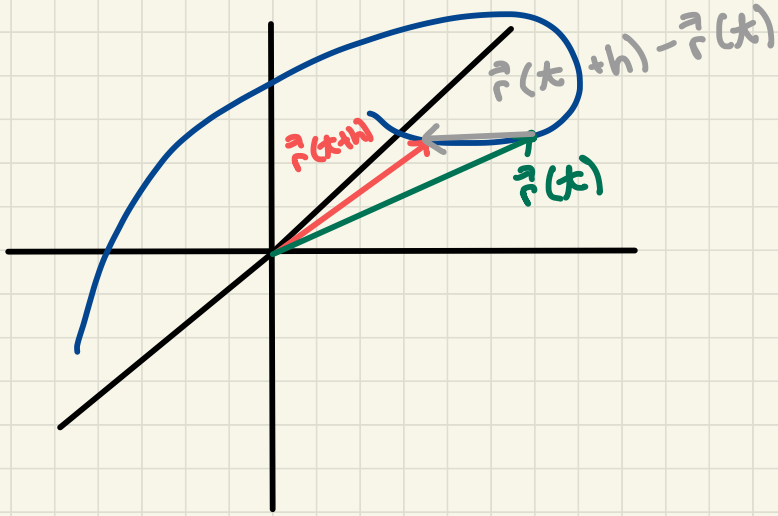


Motion in space

$\vec{r}(t)$ the position vector at time t

$$\vec{v}(t) = \vec{r}'(t) = \lim_{h \rightarrow 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

the velocity vector at time t



velocity is a vector quantity

(magnitude and direction)

magnitude is $|\vec{v}(t)|$, the speed

$\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t)$, the acceleration

Example : the position vector of an object

is given by $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$

find the velocity, speed, and acceleration

Ans :

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is given by $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$

find the velocity, speed, and acceleration

Ans: $\vec{v}(t) = \vec{r}'(t) = \langle -\sin(t), \cos(t), 1 \rangle$

$$|\vec{v}(t)| = \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 1^2}$$

$$= \sqrt{\sin^2(t) + \cos^2(t) + 1} = \sqrt{2}$$

$\vec{a}(t) = \vec{v}'(t) = \langle -\cos(t), -\sin(t), 0 \rangle$

Example: particle starts at an initial position

$\vec{r}(0) = \langle 1, 1, 3 \rangle$ and initial velocity

$\vec{v}(0) = \langle 4, 5, 0 \rangle$ with acceleration

$\vec{a}(t) = \langle t, e^t, \sin(t) \rangle$

Ans:

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$$\vec{a}(t) = \langle t, e^t, \sin(t) \rangle$$

$$\text{Ans: } \vec{v}(t) = \int \vec{a}(t) dt = \langle \frac{1}{2}t^2, e^t, -\cos(t) \rangle + \vec{C}$$

$$\vec{v}(0) = \langle 0, 1, -1 \rangle + \vec{C} = \langle 4, 5, 0 \rangle \Rightarrow \vec{C} = \langle 4, 4, 1 \rangle$$

$$\vec{v}(t) = \langle \frac{1}{2}t^2 + 4, e^t + 4, 1 - \cos(t) \rangle$$

$$\vec{r}(t) = \int \vec{v}(t) dt = \langle \frac{1}{6}t^3 + 4t, e^t + 4t, t - \sin(t) \rangle + \vec{C}$$

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$$\vec{r}(0) = \langle 0, 1, 0 \rangle + \vec{C} = \langle 1, 1, 3 \rangle \Rightarrow \vec{C} = \langle 1, 0, 3 \rangle$$

$$\vec{r}(t) = \left\langle \frac{1}{6}t^3 + 4t + 1, e^t + 4t, t - \sin(t) + 3 \right\rangle$$

In general,

$$\vec{v}(t) = \vec{v}(t_0) + \int_{t_0}^t \vec{a}(u) du$$

$$\vec{r}(t) = \vec{r}(t_0) + \int_{t_0}^t \vec{v}(u) du$$

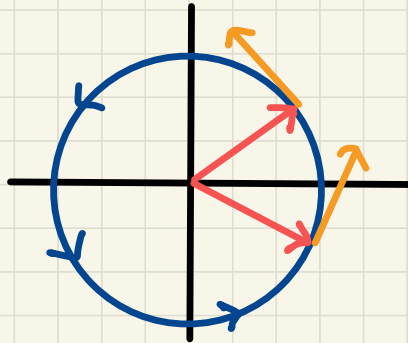
(just a rephrasing of the fundamental theorem of calculus)

Newton's Second Law of Motion

$$\vec{F}(t) = m \vec{a}(t)$$

Example: an object with mass m that moves in a circular path with constant angular speed ω has position vector $\vec{r}(t) = \langle a \cos(\omega t), a \sin(\omega t) \rangle$

find the force acting on the object and show that it is directed towards the origin

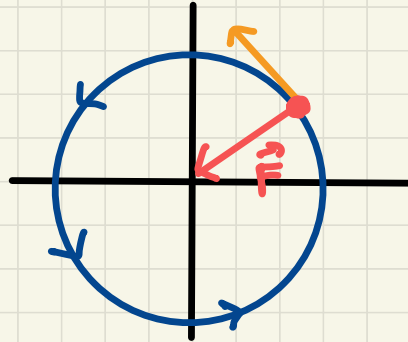
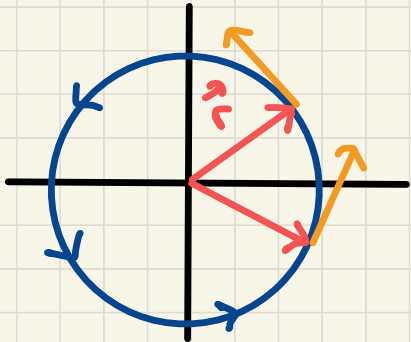


$$\text{Ans: } \vec{r}(t) = \langle a \cos(\omega t), a \sin(\omega t) \rangle$$

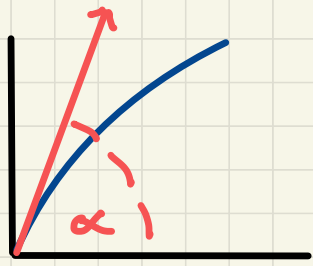
$$\vec{v}(t) = \vec{r}'(t) = \langle -a\omega \sin(\omega t), a\omega \cos(\omega t) \rangle$$

$$\vec{a}(t) = \vec{v}'(t) = \langle -a\omega^2 \cos(\omega t), -a\omega^2 \sin(\omega t) \rangle$$

$$\vec{F}(t) = m\vec{a}(t) = \langle -m a \omega^2 \cos(\omega t), -m a \omega^2 \sin(\omega t) \rangle$$



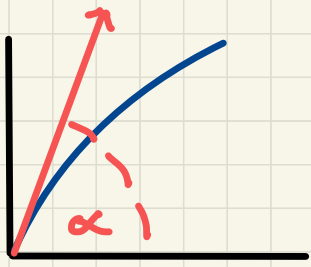
Example: a ball is thrown with an angle of elevation α and initial velocity \vec{v}_0



assume that air resistance is negligible and the only external force is due to gravity

find the position vector of the ball

what value of α maximizes the horizontal distance traveled?

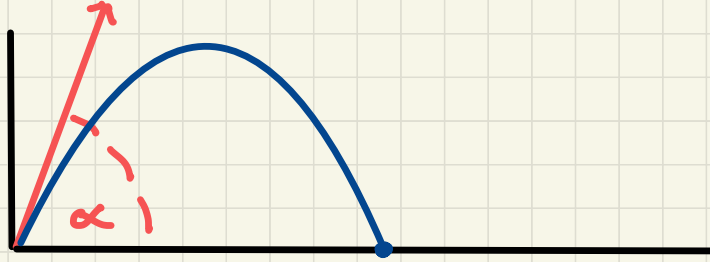
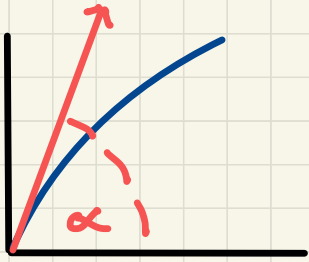


$$\vec{v}(0) = \vec{v}_0 = \langle |\vec{v}_0| \cos(\alpha), |\vec{v}_0| \sin(\alpha) \rangle$$

$$\vec{a}(t) = \langle 0, -g \rangle \quad (g \sim 9.8 \text{ m/s}^2)$$

$$\begin{aligned} \vec{v}(t) &= \vec{v}(0) + \int_0^t \vec{a}(u) du = \vec{v}_0 + \langle 0, -gt \rangle \\ &= \langle |\vec{v}_0| \cos(\alpha), |\vec{v}_0| \sin(\alpha) - gt \rangle \end{aligned}$$

$$\begin{aligned} \vec{r}(t) &= \vec{r}(0) + \int_0^t \vec{v}(u) du = \vec{0} + \langle |\vec{v}_0| \cos(\alpha)t, |\vec{v}_0| \sin(\alpha)t - \frac{gt^2}{2} \rangle \\ &= \langle |\vec{v}_0| \cos(\alpha)t, |\vec{v}_0| \sin(\alpha)t - \frac{gt^2}{2} \rangle \end{aligned}$$



$$\vec{r}(t) = \left\langle |\vec{v}_0| \cos(\alpha) t, |\vec{v}_0| \sin(\alpha) t - \frac{gt^2}{2} \right\rangle$$

$$|\vec{v}_0| \sin(\alpha) t - \frac{gt^2}{2} = 0 \Leftrightarrow t \left(|\vec{v}_0| \sin(\alpha) - \frac{gt}{2} \right) = 0$$

$$\Leftrightarrow t = 0 \text{ or } t = \frac{2|\vec{v}_0| \sin(\alpha)}{g}$$

horizontal distance traveled:
$$\frac{2|\vec{v}_0|^2 \sin(\alpha) \cos(\alpha)}{g} = \frac{|\vec{v}_0|^2 \sin(2\alpha)}{g}$$

$\alpha = \frac{\pi}{4}$ maximizes

Question: how does $|\vec{v}_0|$ compare
to $|\vec{v}(t_0)|$ for

$$t_0 = \frac{2|\vec{v}_0|\sin(\alpha)}{g}$$

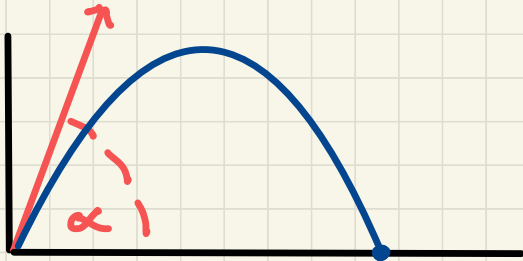
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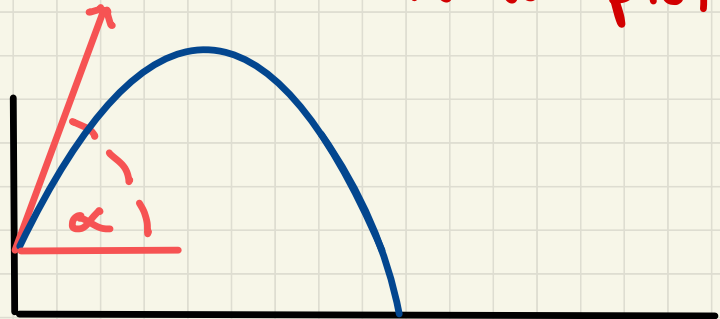
Ans: they're the same!

Example: a cannonball is fired with initial speed 100 m/s at an angle of elevation $\frac{\pi}{4}$ at a position 5 m above ground level. How far does the cannonball travel horizontally before hitting the ground and at what speed?

Old picture



New picture



Old calculation

$$\vec{r}(0) = \langle 0, 0 \rangle$$

$$\vec{v}(0) = \langle |\vec{v}_0| \cos(\alpha), |\vec{v}_0| \sin(\alpha) \rangle$$

$$\vec{a}(t) = \langle 0, -g \rangle$$

New calculation

$$\vec{r}(0) = \langle 0, 5 \rangle$$

$$\begin{aligned} \vec{v}(0) &= \langle 100 \cos\left(\frac{\pi}{4}\right), 100 \sin\left(\frac{\pi}{4}\right) \rangle \\ &= \langle 50\sqrt{2}, 50\sqrt{2} \rangle \end{aligned}$$

$$\vec{a}(t) = \langle 0, -9.8 \rangle$$

$$\vec{v}(t) = \vec{v}(0) + \int_0^t \vec{a}(u) du$$

$$= \langle 50\sqrt{2}, 50\sqrt{2} - 9.8t \rangle$$

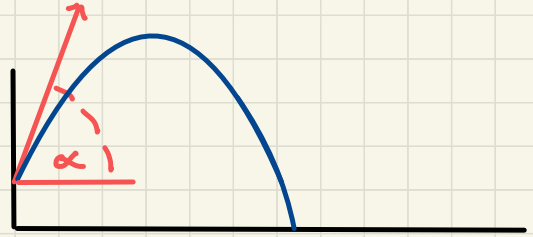
$$\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(u) du$$

$$= \langle 50\sqrt{2}t, 50\sqrt{2}t - \frac{9.8t^2}{2} + 5 \rangle$$

$$\vec{a}(t) = \langle 0, -9.8 \rangle$$

$$\vec{v}(t) = \langle 50\sqrt{2}, 50\sqrt{2} - 9.8t \rangle$$

$$\vec{r}(t) = \langle 50\sqrt{2}t, 50\sqrt{2}t - \frac{9.8t^2}{2} + 5 \rangle$$



$$50\sqrt{2}t - \frac{9.8t^2}{2} + 5 = 0$$

quadratic formula $\Rightarrow t =$

$$\frac{-50\sqrt{2} \pm \sqrt{5000 - 4\left(-\frac{9.8}{2}\right)(5)}}{2\left(-\frac{9.8}{2}\right)}$$

$$= \frac{-50\sqrt{2} \pm \sqrt{5000 + 2(9.8)(5)}}{-9.8}$$

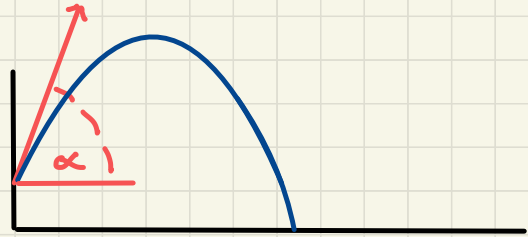
want positive time so

$$t_0 = \frac{50\sqrt{2} + \sqrt{5098}}{9.8}$$

$$\vec{a}(t) = \langle 0, -9.8 \rangle$$

$$\vec{v}(t) = \langle 50\sqrt{2}, 50\sqrt{2} - 9.8t \rangle$$

$$\vec{r}(t) = \langle 50\sqrt{2}t, 50\sqrt{2}t - \frac{9.8t^2}{2} + 5 \rangle$$



$$t_0 = \frac{50\sqrt{2} + \sqrt{5098}}{9.8}$$

$$\vec{r}(t_0) = \left\langle 50\sqrt{2} \frac{50\sqrt{2} + \sqrt{5098}}{9.8}, 0 \right\rangle$$

$$\begin{aligned} |\vec{v}(t_0)| &= |\langle 50\sqrt{2}, -\sqrt{5098} \rangle| = \sqrt{5000 + 5098} \\ &= \sqrt{10098} > 100 \end{aligned}$$