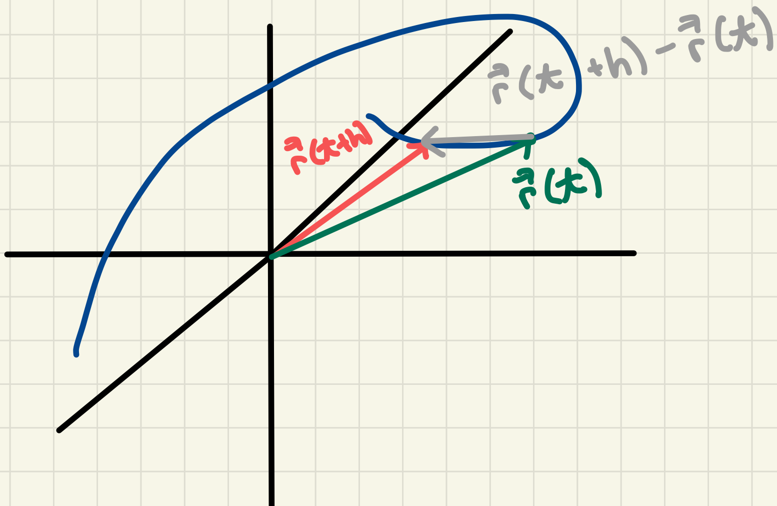
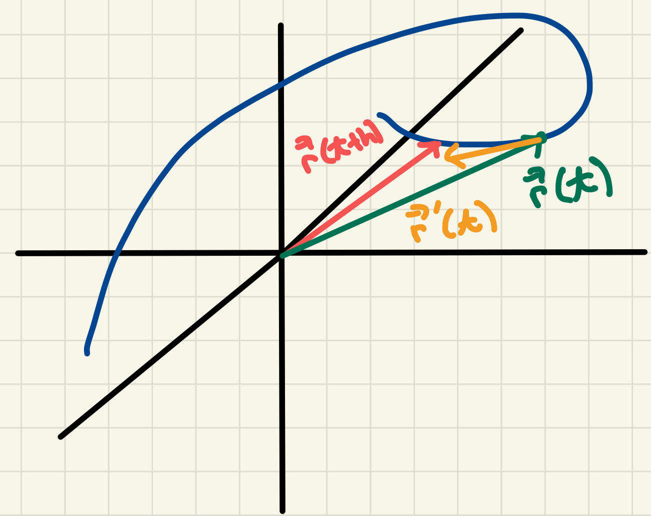
Motion in space

$\vec{r}(t)$ the position vector at time t

$$\vec{v}(t) = \vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

the velocity vector at time t





velocity is a vector quantity

(magnitude and direction)

magnitude is $|\vec{v}(t)|$, the speed

$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$, the acceleration

Example : the position vector of an object is given by $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$ find the velocity, speed, and acceleration

Ans :

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Ans: $\vec{v}(t) = \vec{r}\,'(t) = \langle -\sin(t), \cos(t), 1 \rangle$

$$|\vec{v}(t)| = \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 1^2}$$

$$= \sqrt{\sin^2(t) + \cos^2(t) + 1} = \sqrt{2}$$

$$\vec{a}(t) = \vec{v}\,'(t) = \langle -\cos(t), -\sin(t), 0 \rangle$$

Example: particle starts at an initial position

$\vec{r}(0) = \langle 1, 1, 3 \rangle$ and initial velocity

$\vec{v}(0) = \langle 4, 5, 0 \rangle$ with acceleration

$\vec{a}(t) = \langle t, e^t, \sin(t) \rangle$

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Ans: $\vec{v}(t) = \int \vec{a}(t)\,dt = \langle \frac{1}{2}t^2, e^t, -\cos(t) \rangle + \vec{C}$

$\vec{v}(0) = \langle 0, 1, -1 \rangle + \vec{c} = \langle 4, 5, 0 \rangle \Rightarrow \vec{c} = \langle 4, 4, 1 \rangle$

$$\vec{v}(t) = \langle \frac{1}{2}t^2 + 4, e^t + 4, 1 - \cos(t) \rangle$$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle \frac{1}{6}t^3 + 4t, e^t + 4t, t - \sin(t) \rangle + \vec{C}$$

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$$\vec{r}(0) = \langle 0, 1, 0 \rangle + \vec{C} = \langle 1, 1, 3 \rangle \Rightarrow \vec{C} = \langle 1, 0, 3 \rangle$$

$$\vec{r}(t) = \langle \frac{1}{6}t^3 + 4t + 1, e^t + 4t, t - \sin(t) + 3 \rangle$$

In general,

$$\vec{v}(t) = \vec{v}(t_0) + \int_{t_0}^{t} \vec{a}(u)\, du$$

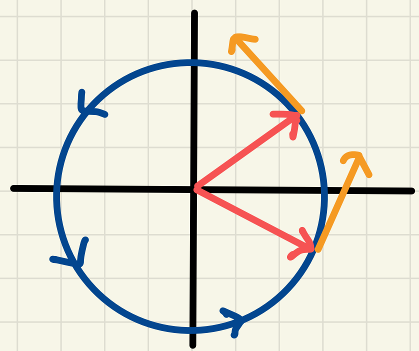
$$\vec{r}(t) = \vec{r}(t_0) + \int_{t_0}^{t} \vec{v}(u)\, du$$

(just a rephrasing of the fundamental theorem of calculus)

Newton's Second Law of Motion

$$\vec{F}(t) = m\vec{a}(t)$$

Example: an object with mass m that moves in a circular path with constant angular speed ω has position vector $\vec{r}(t) = \langle a\cos(\omega t), a\sin(\omega t)\rangle$ find the force acting on the object and show that it is directed towards the origin

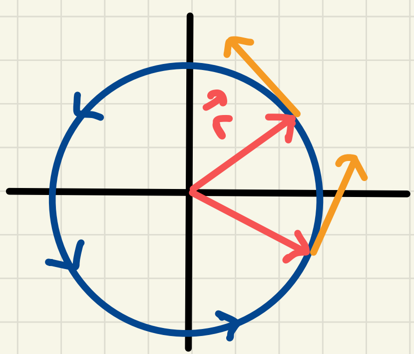


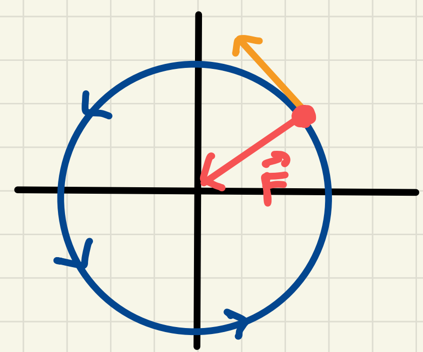
Ans: $\vec{r}(t) = \langle a\cos(\omega t),\ a\sin(\omega t)\rangle$

$$\vec{v}(t) = \vec{r}\,'(t) = \langle -a\omega\sin(\omega t),\ a\omega\cos(\omega t)\rangle$$

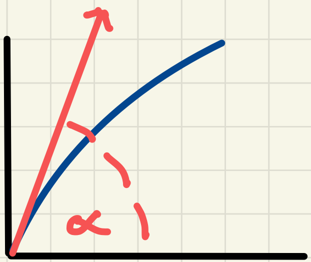
$$\vec{a}(t) = \vec{v}\,'(t) = \langle -a\omega^2\cos(\omega t),\ -a\omega^2\sin(\omega t)\rangle$$

$$\vec{F}(t) = m\vec{a}(t) = \langle -ma\omega^2\cos(\omega t),\ -ma\omega^2\sin(\omega t)\rangle$$





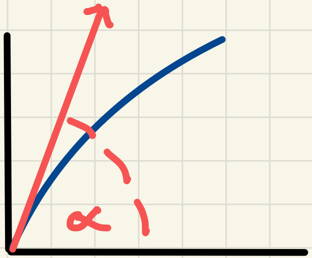
Example: a ball is thrown with an angle of elevation α and initial velocity $\vec{v}_0$



assume that air resistance is negligible and the only external force is due to gravity

find the position vector of the ball

what value of α maximizes the horizontal distance traveled?



$$\vec{v}(0) = \vec{v}_0 = \langle |\vec{v}_0| \cos(\alpha), |\vec{v}_0| \sin(\alpha) \rangle$$

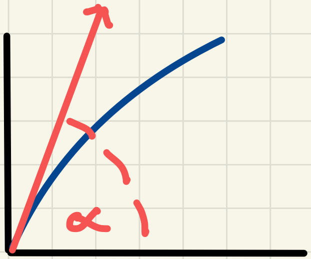
$$\vec{a}(t) = \langle 0, -g \rangle \qquad (g \sim 9.8 \, m/s^2)$$

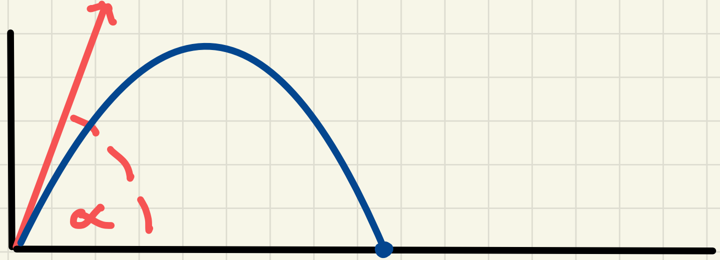
$$\vec{v}(t) = \vec{v}(0) + \int_0^t \vec{a}(u)\, du = \vec{v}_0 + \langle 0, -gt \rangle$$

$$= \langle |\vec{v}_0| \cos(\alpha), |\vec{v}_0| \sin(\alpha) - gt \rangle$$

$$\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(u)\, du = \vec{0} + \langle |\vec{v}_0| \cos(\alpha) t, |\vec{v}_0| \sin(\alpha) t - \frac{gt^2}{2} \rangle$$

$$= \langle |\vec{v}_0| \cos(\alpha) t, |\vec{v}_0| \sin(\alpha) t - \frac{gt^2}{2} \rangle$$





$$\vec{r}(t) = \left\langle |\vec{v}_0|\cos(\alpha)\,t,\ |\vec{v}_0|\sin(\alpha)\,t - \frac{gt^2}{2} \right\rangle$$

$$|\vec{v}_0|\sin(\alpha)t - \frac{gt^2}{2} = 0 \iff t\left(|\vec{v}_0|\sin(\alpha) - \frac{gt}{2}\right) = 0$$

$$\iff t = 0 \quad \text{or} \quad t = \frac{2|\vec{v}_0|\sin(\alpha)}{g}$$

horizontal distance traveled: $\dfrac{2|\vec{v}_0|^2\sin(\alpha)\cos(\alpha)}{g} = \dfrac{|\vec{v}_0|^2\sin(2\alpha)}{g}$

$\alpha = \frac{\pi}{4}$ maximizes

Question: how does $|\vec{v}_0|$ compare

to $|\vec{v}(t_0)|$ for

$$t_0 = \frac{2|\vec{v}_0|\sin(\alpha)}{g}$$

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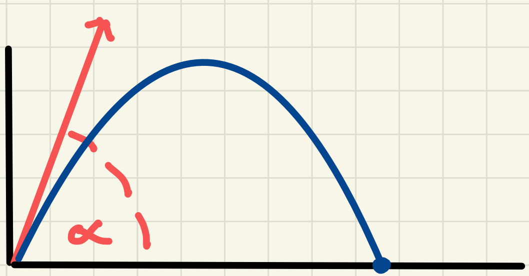
Ans: they're the same!

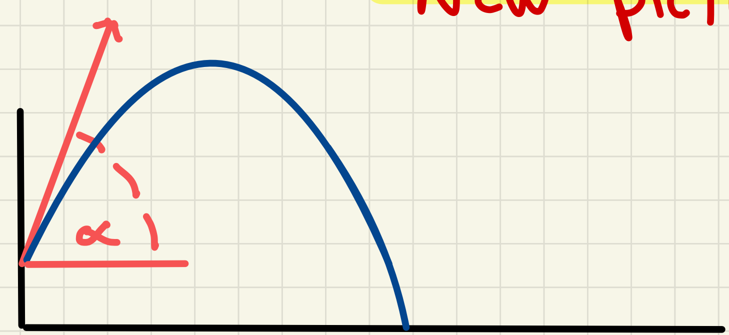
Example: a cannonball is fired with initial speed 100 m/s at an angle of elevation $\frac{\pi}{4}$ at a position 5 m above ground level

how far does the cannonball travel horizontally before hitting the ground and at what speed?

Old picture



New picture



Old calculation

$\vec{r}(0) = \langle 0, 0 \rangle$

$\vec{v}(0) = \langle |\vec{v}_0| \cos(\alpha), |\vec{v}_0| \sin(\alpha) \rangle$

$\vec{a}(t) = \langle 0, -g \rangle$

New calculation

$\vec{r}(0) = \langle 0, 5 \rangle$

$\vec{v}(0) = \langle 100 \cos(\frac{\pi}{4}), 100 \sin(\frac{\pi}{4}) \rangle$

$= \langle 50\sqrt{2}, 50\sqrt{2} \rangle$

$\vec{a}(t) = \langle 0, -9.8 \rangle$

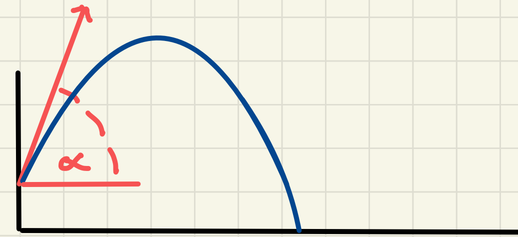
$\vec{v}(t) = \vec{v}(0) + \int_0^t \vec{a}(u)\,du$

$= \langle 50\sqrt{2}, 50\sqrt{2} - 9.8t \rangle$

$\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(u)\,du$

$= \langle 50\sqrt{2}\,t, 50\sqrt{2}\,t - \frac{9.8t^2}{2} + 5 \rangle$

$\vec{a}(t) = \langle 0, -9.8 \rangle$



$\vec{v}(t) = \langle 50\sqrt{2}, 50\sqrt{2} - 9.8t \rangle$

$\vec{r}(t) = \langle 50\sqrt{2}t, 50\sqrt{2}t - \frac{9.8t^2}{2} + 5 \rangle$

$$50\sqrt{2}t - \frac{9.8t^2}{2} + 5 = 0$$

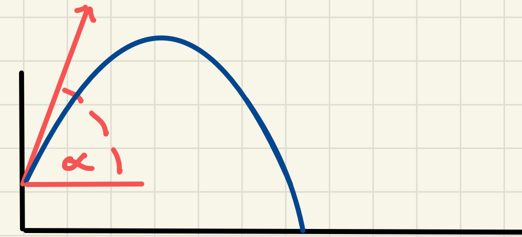
quadratic formula $\Rightarrow$ $t = \dfrac{-50\sqrt{2} \pm \sqrt{5000 - 4\left(-\frac{9.8}{2}\right)(5)}}{2\left(-\frac{9.8}{2}\right)}$

$$= \frac{-50\sqrt{2} \pm \sqrt{5000 + 2(9.8)(5)}}{-9.8}$$

want positive time so

$$t_0 = \frac{50\sqrt{2} + \sqrt{5098}}{9.8}$$

$$\vec{a}(t) = \langle 0, -9.8 \rangle$$



$$\vec{v}(t) = \langle 50\sqrt{2}, 50\sqrt{2} - 9.8t \rangle$$

$$\vec{r}(t) = \langle 50\sqrt{2}t, 50\sqrt{2}t - \frac{9.8t^2}{2} + 5 \rangle$$

$$t_0 = \frac{50\sqrt{2} + \sqrt{5098}}{9.8}$$

$$\vec{r}(t_0) = \langle 50\sqrt{2}\,\frac{50\sqrt{2} + \sqrt{5098}}{9.8}, 0 \rangle$$

$$|\vec{v}(t_0)| = |\langle 50\sqrt{2}, -\sqrt{5098} \rangle| = \sqrt{5000 + 5098}$$

$$= \sqrt{10098} > 100$$