Partial derivatives

What you've seen in previous classes:

$$f: \mathbb{R} \to \mathbb{R}$$
$$x \mapsto f(x)$$

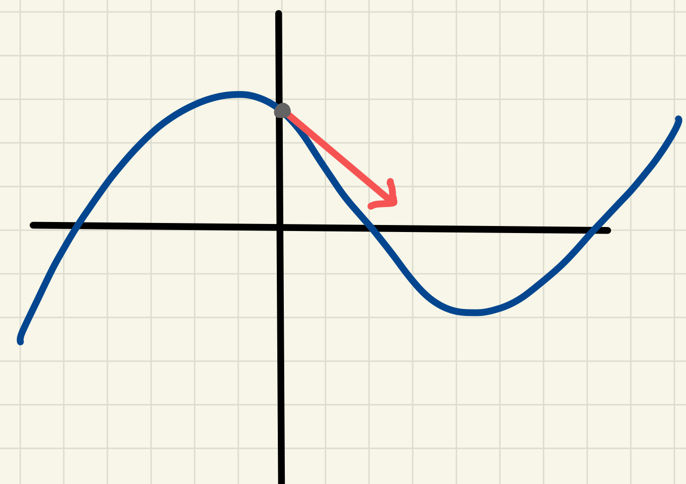
e.g. $x^2 + e^x$

, $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

alternative notation: $f_x(a)$

Geometric interpretation:

"instantaneous rate of change"



Today: $\quad f: \mathbb{R}^2 \to \mathbb{R}$

$$(x,y) \mapsto f(x,y)$$

e.g. $\cos(xy) + e^{x+y}$

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

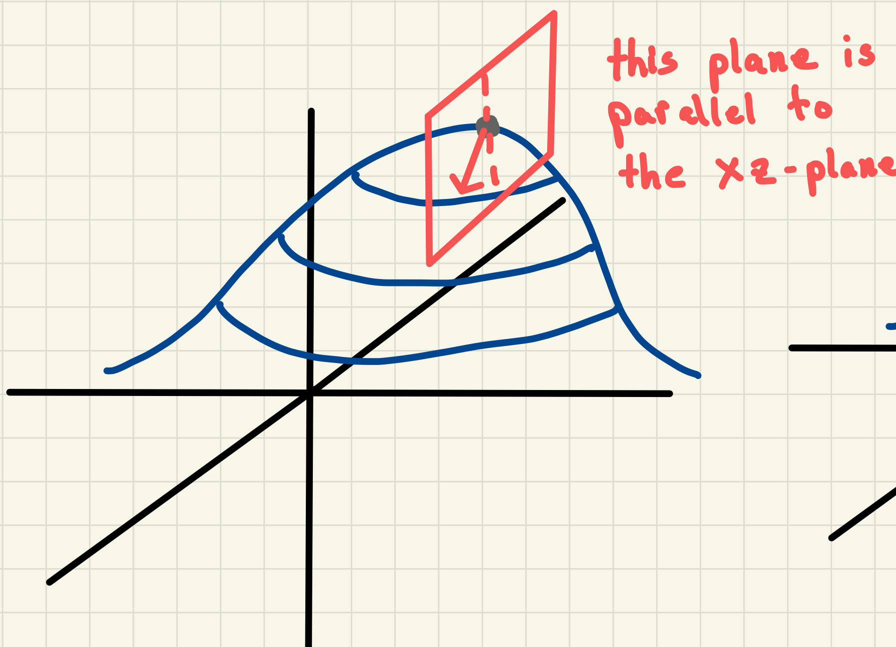
partial derivative with respect to x at (a,b)

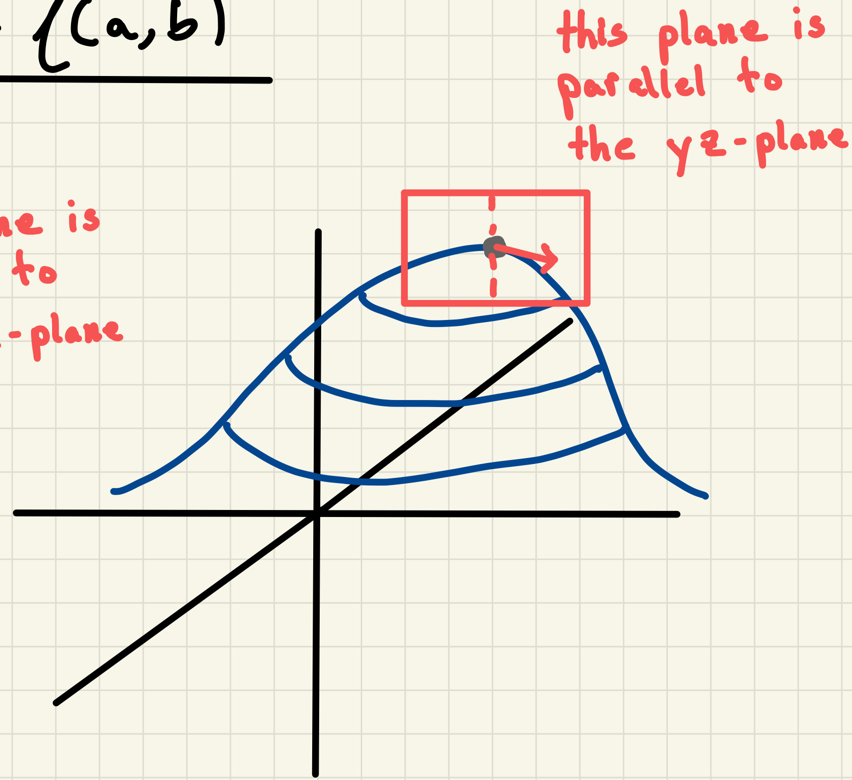
$$f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

partial derivative with respect to y at (a,b)

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$$





Notation

if $z = f(x,y)$, we write

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = f_1 = D_1 f = D_x f$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = f_2 = D_2 f = D_y f$$

Question : how to calculate

Question: how to calculate

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Answer:

to calculate f_x,

treat y as a constant

and differentiate f

with respect to x

to calculate f_y,

treat x as a constant

and differentiate f

with respect to y

Example: let $f(x,y) = \frac{x}{y} + e^{xy^2} + \sin(x^2 y)$

find $f_x\left(\sqrt{\frac{\pi}{2}}, 1\right)$ and $f_y\left(\sqrt{\frac{\pi}{2}}, 1\right)$

Ans:

Example: let $f(x,y) = \frac{x}{y} + e^{xy^2} + \sin(x^2 y)$

find $f_x\left(\sqrt{\frac{\pi}{2}}, 1\right)$ and $f_y\left(\sqrt{\frac{\pi}{2}}, 1\right)$

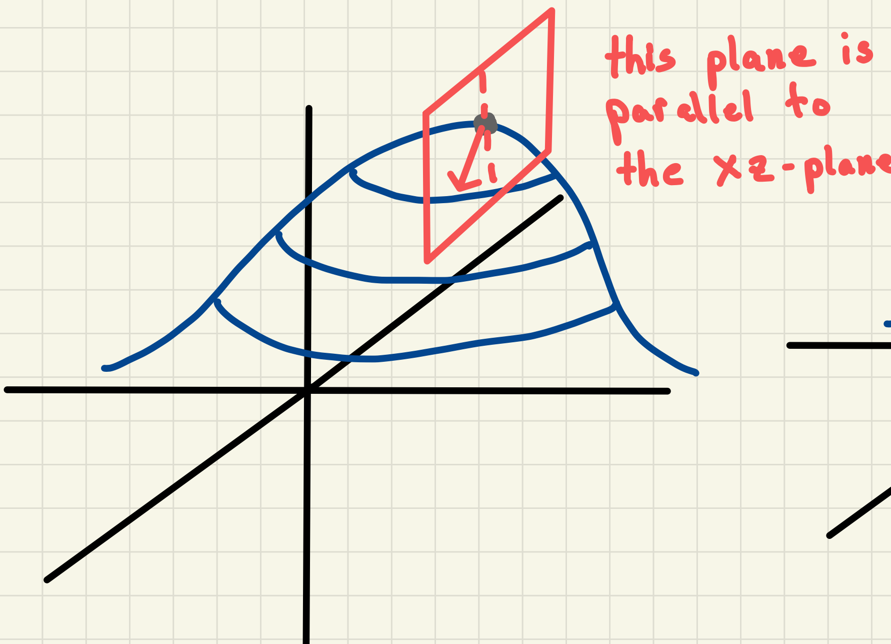
Ans: $f_x(x,y) = \frac{1}{y} + e^{xy^2}(y^2) + \cos(x^2 y)(2xy)$

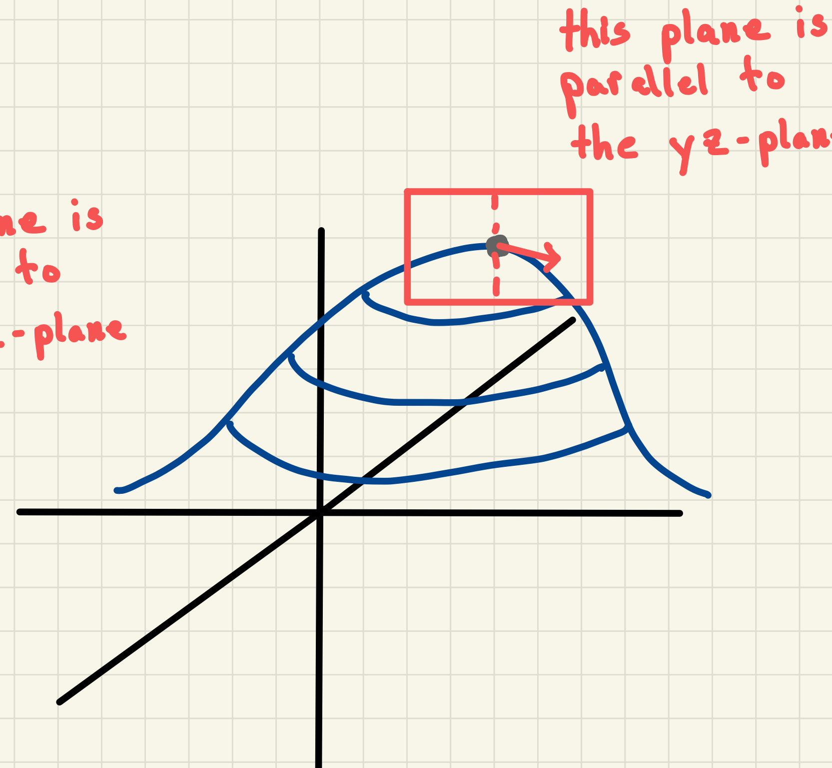
$f_x\left(\sqrt{\frac{\pi}{2}}, 1\right) = 1 + e^{\sqrt{\frac{\pi}{2}}} + 0$

$f_y(x,y) = \frac{-x}{y^2} + e^{xy^2}(2xy) + \cos(x^2 y)(x^2)$

$f_y\left(\sqrt{\frac{\pi}{2}}, 1\right) = -\sqrt{\frac{\pi}{2}} + e^{\sqrt{\frac{\pi}{2}}}\left(\sqrt{2\pi}\right) + 0$

Remember the geometric interpretation!





Example: if $f(x,y) = \ln\left(\frac{x}{1+xy}\right)$,

calculate f_x and f_y

Ans:

Example: if $f(x,y) = \ln\left(\frac{x}{1+xy}\right)$,

calculate f_x and f_y

Ans: $$f_x(x,y) = \frac{1}{\frac{x}{1+xy}} \frac{(1+xy) - xy}{(1+xy)^2} = \frac{1}{x(1+xy)}$$

$$f_y(x,y) = \frac{1}{\frac{x}{1+xy}} \frac{-x^2}{(1+xy)^2} = \frac{-x}{1+xy}$$

Example (implicit partial differentiation):

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is defined implicitly as a function of x and y by the eqn

$$e^{xy} + \sin(x+y) + xyz + z^3 = \pi$$

Ans:

Example (implicit partial differentiation):

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is defined implicitly as a function of x and y by the eqn

$$e^{xy} + \sin(x+y) + xyz + z^3 = \pi$$

Ans: $\frac{\partial}{\partial x}\left[e^{xy} + \sin(x+y) + xyz + z^3\right] = \frac{\partial}{\partial x}[\pi] = 0$

$$\parallel$$

$$e^{xy}y + \cos(x+y) + \left(yz + xy\frac{\partial z}{\partial x}\right) + 3z^2\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{-e^{xy}y - \cos(x+y) - yz}{xy + 3z^2}$$

Ans (cont):

$$\frac{\partial}{\partial y}\left[e^{xy} + \sin(x+y) + xyz + z^3\right] = \frac{\partial}{\partial y}[\pi] = 0$$

$$\shortparallel$$

$$e^{xy}x + \cos(x+y) + \left(xz + yx\frac{\partial z}{\partial y}\right) + 3z^2\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = \frac{-e^{xy}x - \cos(x+y) - xz}{yx + 3z^2}$$

Functions of more than two variables

if $f(x,y,z)$, can define

$$f_x \quad , \quad f_y \quad , \quad f_z(x,y,z) = \lim_{h\to 0} \frac{f(x,y,z+h) - f(x,y,z)}{h}$$

if $f(x_1, x_2, \ldots, x_n)$, can define

$$f_{x_1} \;, \; f_{x_2} \;, \cdots , \; f_{x_n}(x_1,\ldots,x_n) = \lim_{h\to 0} \frac{f(x_1,\ldots,x_{n-1},x_n+h) - f(x_1,\ldots,x_n)}{h}$$

Example: $f(x,y,z) = \cos(x+yz)e^{(x^2)}$

find f_x, f_y, and f_z

Ans:

Example: $f(x,y,z) = \cos(x+yz)e^{(x^2)}z$

find f_x, f_y, and f_z

Ans: $f_x(x,y,z) = \cos(x+yz)e^{(x^2)}(2x)z - \sin(x+yz)e^{(x^2)}z$

$f_y(x,y,z) = (-\sin(x+yz))e^{(x^2)}z^2$

$f_z(x,y,z) = \cos(x+yz)e^{(x^2)} - \sin(x+yz)(y)e^{(x^2)}z$

Higher derivatives

consider $f(x,y)$

then $f_x(x,y)$ is another function that

we can take a partial derivative of

$$\frac{\partial}{\partial x} f_x(x,y) = f_{xx}(x,y) \qquad \frac{\partial}{\partial y} f_x(x,y) = f_{xy}(x,y)$$

second partial derivatives

same idea for f_{yx}, f_{yy}, f_{yxyyxy}

Example: find the second partial derivatives of $f(x,y) = \ln\left(\frac{x}{1+xy}\right)$ (recall that

$$f_x(x,y) = \frac{1}{x(1+xy)} \quad \text{and} \quad f_y(x,y) = \frac{-x}{1+xy}$$)

Ans:

Example: Find the second partial derivatives of $f(x,y) = \ln\left(\frac{x}{1+xy}\right)$ (recall that $f_x(x,y) = \frac{1}{x(1+xy)}$ and $f_y(x,y) = \frac{-x}{1+xy}$)

Ans:

$$f_{xx}(x,y) = \frac{-(1+2xy)}{(x(1+xy))^2}$$

$$f_{yy}(x,y) = \frac{x^2}{(1+xy)^2}$$

$$f_{xy}(x,y) = \frac{-x^2}{(x(1+xy))^2}$$

same

$$f_{yx}(x,y) = \frac{-(1+xy)+xy}{(1+xy)^2}$$

Clairaut's theorem

Suppose f is defined on a disk D that contains the point (a,b). If the functions f_{xy} and f_{yx} are continuous on D, then $f_{xy}(a,b) = f_{yx}(a,b)$

"order doesn't matter (in certain cases)"

also true for higher derivatives:

for example, $f_{xxy} = f_{yxx} = f_{xyx}$ (under similar assumptions)

Example: calculate f_{xxyz} if $f(x,y,z) = e^{xy+z^2}$

Ans:

Example: calculate f_{xxyz} if $f(x,y,z) = e^{xy+z^2}$

Ans: $f_x = e^{xy+z^2}\,y$

$$f_{xx} = e^{xy+z^2}\,y^2$$

$$f_{xxy} = e^{xy+z^2}(2y) + e^{xy+z^2}(x)(y^2)$$

$$f_{xxyz} = e^{xy+z^2}(2y)(2z) + e^{xy+z^2}(2z)(x)(y^2)$$

Partial differential eqns

A PDE is an equation relating the partial derivatives of a function

Example: $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

A solution to a PDE is a function whose partial derivatives satisfy the PDE

Example: $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

Show that $u(x,y) = e^x \sin(y)$ is a solution

Ans:

Example: $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

Show that $u(x,y) = e^x \sin(y)$ is a solution

Ans: $u_x(x,y) = e^x \sin(y)$ $\quad$ $u_y(x,y) = e^x \cos(y)$

$u_{xx}(x,y) = e^x \sin(y)$ $\quad$ $u_{yy}(x,y) = -e^x \sin(y)$

$u_{xx} + u_{yy} = 0$ ✓

another solution?

Example: verify that $u(x,t) = \sin(x-at)$ is a solution to the PDE $\dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2}$

Ans:

Example: verify that $u(x,t) = \sin(x-at)$ is a solution to the PDE $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$

Ans: $u_t(x,t) = \cos(x-at)(-a)$

$u_{tt}(x,t) = -\sin(x-at)(-a)(-a) = -a^2 \sin(x-at)$

$u_x(x,t) = \cos(x-at)$

$u_{xx}(x,t) = -\sin(x-at)$

$u_{tt} = a^2 u_{xx}$ ✓