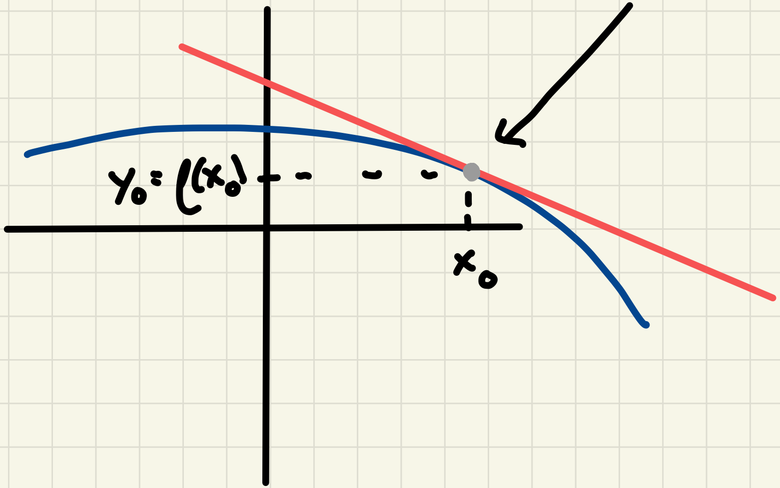
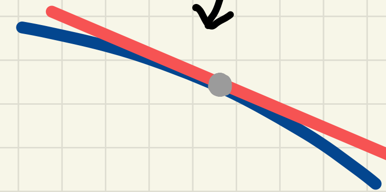
Tangent planes and linear approximations

Before: tangent line

zoom in here and we see

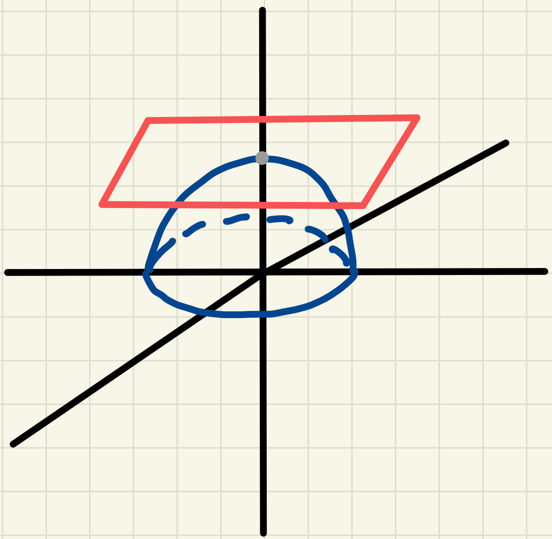




$$y - y_0 = f_x(x_0)(x - x_0)$$

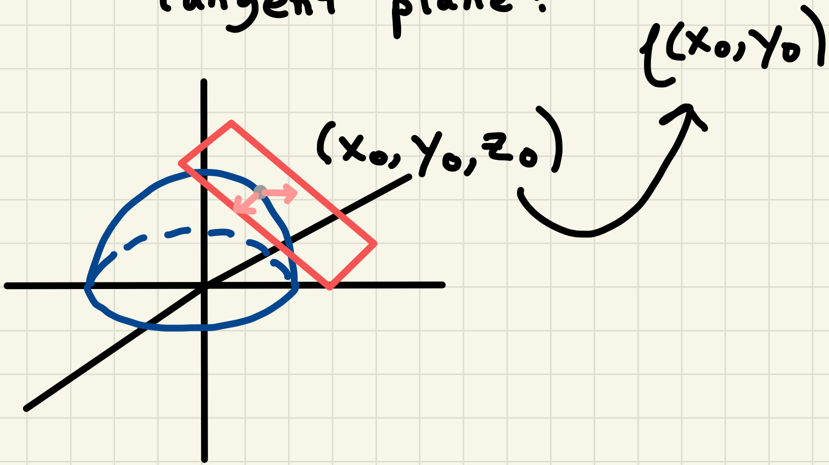
Moral: we can approximate a function by a linear function

Today : tangent plane



Moral : we can approximate a function by a linear function (note: we assume our function has continuous partial derivatives)

How to find the tangent plane?



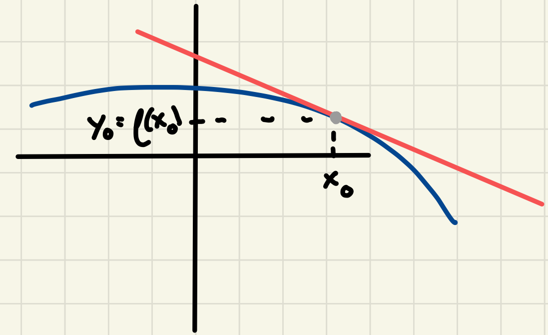
$$\langle 1, 0, f_x(x_0,y_0)\rangle \times \langle 0, 1, f_y(x_0,y_0)\rangle$$

$$= \langle -f_x(x_0,y_0), -f_y(x_0,y_0), 1\rangle$$

$$-f_x(x_0,y_0)(x-x_0) - f_y(x_0,y_0)(y-y_0) + (z-z_0) = 0$$

$$\Rightarrow \quad z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

Tangent line



$$y - y_0 = f_x(x_0)(x - x_0)$$

Example: find the tangent plane to

$z = 3x^2 + 2y^2$ at $(1,1)$

Ans:

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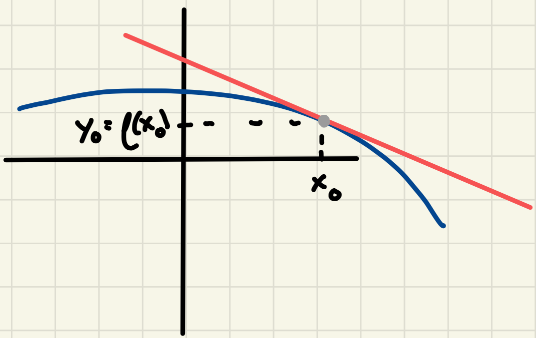
Ans: $x_0 = 1$, $y_0 = 1$, $z_0 = 5$

$f_x(x,y) = 6x$, $f_y(x,y) = 4y$

$f_x(1,1) = 6$ $\qquad$ $f_y(1,1) = 4$

$$z - 5 = 6(x-1) + 4(y-1)$$

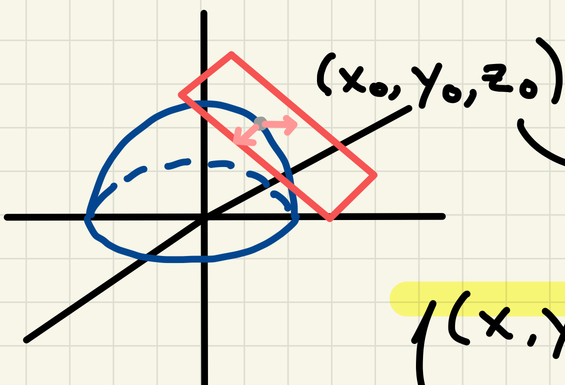
Linear approximations



$$f(x) \approx f_x(x_0)(x - x_0) + y_0$$

for x near x_0

tangent line approximation



$f(x_0, y_0)$

tangent plane approximation
"linearization"

$$f(x, y) \approx f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + z_0$$

for (x, y) near (x_0, y_0)

Example: approximate the function

$z = 3x^2 + 2y^2$ at $(.9, 1.1)$

Ans:

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$z = 3x^2 + 2y^2$ at $(.9, 1.1)$

Ans: tangent plane at $(1,1)$ is

$z - 5 = 6(x-1) + 4(y-1) \Rightarrow z = 6(x-1) + 4(y-1) + 5$

our function $3x^2 + 2y^2 \approx 6(x-1) + 4(y-1) + 5$

near $(1,1)$

actually 4.85

so $3(.9)^2 + 2(1.1)^2 \approx 6(-.1) + 4(.1) + 5 = 4.8$

Differentials

let $y = f(x)$

suppose x changes from a to $a + \Delta x$

the $\Delta y = f(a + \Delta x) - f(a)$

is the corresponding change in y

tangent line $y - f(a) = f_x(a)(x - a)$

says $\Delta y \approx f_x(a)\Delta x$

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tangent line $y - f(a) = f_x(a)(x-a)$

says $\Delta y \approx f_x(a)\Delta x$

let $z = f(x,y)$

suppose x changes from a to $a + \Delta x$

and y changes from b to $b + \Delta y$

tangent plane $z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$

says $\Delta z \approx f_x(a,b)\Delta x + f_y(a,b)\Delta y$

for $z = f(x, y)$

$$\Delta z \approx f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y$$

we define the differentials dx and dy

the differential dz is called the total diffential and is defined by

$$dz = f_x(x,y)\,dx + f_y(x,y)\,dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$z = f(x, y)$$

$$\Delta z \approx f_x(a, b)\Delta x + f_y(a, b)\Delta y$$

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$\Delta z \approx dz$$

Example: if $z = f(x, y) = e^x + 2xy^2$,

find the differential dz

if x changes from 0 to $.01$

and y changes from 1 to $.99$

compare Δz and dz

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Ans: $dz = f_x(x,y)\,dx + f_y(x,y)\,dy = (e^x + 2y^2)\,dx + (4xy)\,dy$

$\Delta z = f(.01, .99) - f(0,1) = e^{.01} + 2(.01)(.99)^2 - 1 \approx .0297$

$dz = (1+2).01 + (4 \cdot 0 \cdot 1)(-.01) = .03$

$(x,y) = (0,1)$

Example: the base radius and height of a cylinder are measured as 5 cm and 10 cm respectively, with a possible error of as much as .1 cm in each dimension

use differentials to estimate the maximum error in the calculated volume of the cylinder

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Ans: $z = f(x,y) = \pi x^2 y$, $dz = 2\pi xy\,dx + \pi x^2\,dy$

$dz = 100\pi(.1) + 25\pi(.1) = 12.5\pi$

Functions of more than two variables

single variable $y = f(x)$

two variables $z = f(x, y)$

multivariable $w = f(x, y, z)$

(or $y = f(x_1, x_2, \ldots, x_n)$)

$$dy = f_x \, dx$$

$$dz = f_x \, dx + f_y \, dy$$

$$dw = f_x \, dx + f_y \, dy + f_z \, dz$$

$$(dy = f_{x_1} dx_1 + f_{x_2} dx_2 + \cdots + f_{x_n} dx_n)$$

$\downarrow$

still a linear approximation to Δy

Example: the dimensions of a rectangular box are measured to be 10 cm, 20 cm, and 50 cm with a margin of error of .1cm in each dimension

use differentials to estimate the largest possible error when the volume of the box is calculated

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Ans: $V = \ell w h$ $\qquad$ $dV = wh\,d\ell + \ell h\,dw + \ell w\,dh$

$$dV = 1000\,d\ell + 500\,dw + 200\,dh = 100 + 50 + 20 = 170$$