MATH 180A: Introduction to Probability

Lecture B00 (Nemish)

www.math.ucsd.edu/~ynemish/teaching/180a

Lecture C00 (Au)

www.math.ucsd.edu/~bau/f20.180a

Today: Definition of Probability.
Random sampling.

Next: ASV 1.2

Video: Prof. Todd Kemp, Fall 2019

Week 0/1:

- Homework 0 (due Wednesday October 7)
- Homework 1 (due Friday October 9)
- Join Piazza

1.1

The world around us is fundamentally random.

* 1600s – games of chance
* 1900s – quantum theory
* 1950s – finance / insurance
* 1980s – chemical reactions inside our cells
* 2000s – complex networks; machine learning

The modern rigorous foundation of probability theory goes back to 1933 Kolmogorov

Ingredients

Sample Space Ω = the set of possible outcomes in an experiment. $\{HH, HT, TH, TT\}$

Events $\mathcal{F}$ = collections of outcomes. $E = \{\text{at least } 1\ H\}$
$= \{HH, HT, TH\}$

Probability Measure $\mathbb{P} : \mathcal{F} \to [0, 1]$

Kolmogorov's Axioms

(i) For any event $A \subseteq \mathcal{F}$, $0 \leq \mathbb{P}(A) \leq 1$.

(ii) $\mathbb{P}(\Omega) = 1$ $\qquad \mathbb{P}(\phi) = 0$

(iii) If $A_1, A_2, A_3, \ldots$ are disjoint events, then

$$\mathbb{P}(A_1 \cup A_2 \cup A_3 \cup \cdots) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) + \cdots$$

$$\mathbb{P}(E \cup T) = \frac{2}{3}$$

$$\shortparallel$$

$$\mathbb{P}(E) + \mathbb{P}(T)$$

$$\frac{1}{2} + \frac{1}{3}$$

E.g. A fair die. $\Omega = \{1, 2, 3, 4, 5, 6\}$ $\qquad \mathbb{P}(\{1\}) = \mathbb{P}(\{2\}) = \cdots = \mathbb{P}(\{6\}) = \frac{1}{6}$

E = even, O = odd $\qquad E = \{2, 4, 6\}$ $\quad O = \{1, 3, 5\}$

T = divisible by 3 $\qquad T = \{3, 6\}$

$E \cup O = \Omega$

$E \cup T = \{2, 3, 4, 6\}$

$\mathbb{P}(E) = \frac{3}{6} = \frac{1}{2}$

$\mathbb{P}(T) = \frac{2}{6} = \frac{1}{3}$.

$\mathbb{P}(E \cup T)$

$= \mathbb{P}\{2, 3, 4, 6\}$

$= \mathbb{P}\{2\} + \mathbb{P}\{3\} + \mathbb{P}\{4\} + \mathbb{P}\{6\}$

$= 4 \cdot \frac{1}{6} = \frac{2}{3}$.

Probability Space $(\Omega, \mathcal{F}, \mathbb{P})$

sample space $\nearrow$ Ω events $\uparrow$ $\mathcal{F}$ probability measure $\nwarrow$ $\mathbb{P}$

Determined by the <u>experiment</u> being <u>modeled</u>

Key property: if $\underbrace{A_1, A_2, A_3, \ldots}_{\in \mathcal{F}}$ are disjoint $(A_i \cap A_j = \phi \;\; i \neq j)$

then $\mathbb{P}(A_1 \cup A_2 \cup A_3 \cup \cdots) = \sum_j \mathbb{P}(A_j)$

Important special case: if $\#\Omega < \infty$ then

$$\Omega = \{\omega_1, \omega_2, \omega_3, \ldots, \omega_n\}$$

the singleton sets $\{\omega_1\}, \{\omega_2\}, \ldots, \{\omega_n\}$ are disjoint

$\therefore$ $\mathbb{P}(A) = \{a_1\} \cup \{a_2\} \cup \cdots \cup \{a_r\}$ $\mathbb{P}(A) = \sum_{j=1}^{r} \mathbb{P}(\{a_j\})$

$A = \{a_1, a_2, \ldots, a_r\}$

Uniform Probability Measure (Sampling)

If Ω is finite, the **uniform probability measure** is defined by:

$$\text{For each } \omega \in \Omega, \quad \mathbb{P}(\{\omega\}) = \frac{1}{\#\Omega}$$

$$\Rightarrow \text{For any event } A, \quad \mathbb{P}(A) = \frac{\#A}{\#\Omega}$$

This means calculating probabilities in such models is tantamount to <u>Counting</u>.

E.g. A fair die is cast 2 times. What is the probability that the sum is 4?

$$\Omega = \{ (i,j) : 1 \le i,j \le 6 \} \qquad \#\Omega = 36$$

$$A = \{ (1,3), (2,2), (3,1) \} \qquad \#A = 3$$

(sum is 4)

$$\mathbb{P}(A) = \frac{3}{36} = \frac{1}{12} \simeq 8.3\%$$

E.g. A fair coin is tossed 3 times.

$A = \{\text{at least two tails}\}$

$B = \{\text{exactly two tails}\}$

$\Omega = \{HHH, HHT, \dots, TTT\}$ $\quad \#\Omega = 8$

$= \{(i, j, k) : \quad i, j, k \in \{T, H\}\}$

$A = \{TTH, THT, HTT, TTT\}$ $\quad \mathbb{P}(A) = \frac{\#A}{8} = \frac{4}{8} = \frac{1}{2}$

$B = \{TTH, THT, HTT\}$ $\quad \mathbb{P}(B) = \frac{3}{8}.$

Think Pair Share

There are 10 people on a committee.
How many different ways are there to select a subcommittee of 4 people?

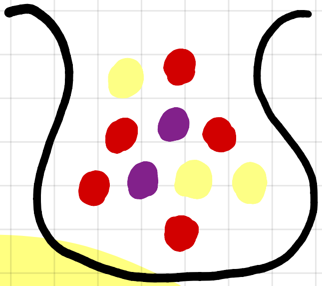
(a) $10 \cdot 10 \cdot 10 \cdot 10 = 10^4 = 10{,}000$

(b) $10 \cdot 9 \cdot 8 \cdot 7 = 5{,}040$

(c) $\binom{10}{4} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4!} = 210$

(d) $\frac{10!}{4!} = 151{,}200$

Combinatorics



A collection of labeled balls $\{1, 2, \dots, n\}$ are in an urn. k are taken out one by one

* with replacement, or
* without replacement $(k \le n)$

$$\Omega = \{(b_1, \dots, b_k) : 1 \le b_j \le n\} = \{1, \dots, n\}^k$$

$b_i \ne b_j$ if $i \ne j$

They are lined up

* in the order they came out, or
* disregarding order.

$$\Omega = \{\{b_1, \dots, b_k\} : \quad \}$$

How many ways?

* with replacement : n^k
* without:
  - & ordered : $n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)$
  - not : $\dfrac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \dfrac{n!}{k!\,(n-k)!} = \dbinom{n}{k}$

when $k=n$ $\Downarrow$
$n!$

Sampling With Replacement

Toss a fair coin n times; record a statistic observing # H vs. # T.

E.g. $n=10$, $\mathbb{P}\{\text{odd rolls are all } H\}$.

$$\Omega = \{(c_1, c_2, c_3, \ldots, c_{10}) : \text{"}\forall j\text{"}\; c_j \in \{H,T\}\}$$

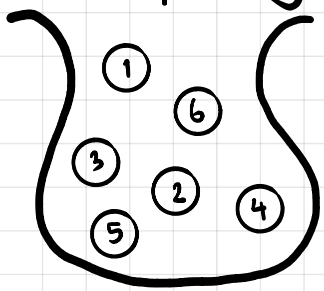
$$\#\Omega = 2^{10}$$

$$A = \{c_1 = c_3 = c_5 = c_7 = c_9 = H\}$$

$$= \{(H,*,H,*,H,*,H,*,H,*)\} \qquad \#A = 2^5$$

$$\mathbb{P}(A) = \frac{2^5}{2^{10}} = \frac{1}{2^5} = \frac{1}{32}.$$

Sampling without Replacement (order matters)



There are 6 labeled balls in an urn. 3 are removed in sequence (without replacement), and lined up in order.

What is the probability that the first two are (3, 6)?

$$\Omega = \{(b_1, b_2, b_3) : \quad 1 \le b_j \le 6, \quad b_1 \ne b_2,\ b_2 \ne b_3,\ b_1 \ne b_3\}$$

$$\#\Omega = 6 \cdot 5 \cdot 4 = 120$$

$$A = \{(3, 6, \underbrace{*}_{\in \{1,2,4,5\}})\} \qquad \#A = 4$$

$$\mathbb{P}(A) = \frac{4}{120} = \frac{1}{30}.$$