Today: ASV 3.3 (Expected value)

Next: ASV 3.4, 3.5

Week 4: Quiz 3 (next Wednesday, Nov 4)
Homework 4 (due next Friday, Nov 6)
**Expectation**

**Definition**: Let $X$ be a discrete random variable with possible values $t_1, t_2, t_3, \ldots$. The expectation or expected value of $X$ is

$$E(X) := \sum_j t_j P(X = t_j)$$

It is also called the **mean** of $X$, and is often denoted $\mu$. 

E.g. Let $X$ be a discrete random variable with probability mass fn.

$$P_X(k) = \frac{c}{k(k+1)}, \quad k = 1, 2, 3, \ldots$$

$$\sum_{k=1}^{\infty} P_X(k) = 1$$

Find $c$, and compute $E(X)$.

$$1 = \sum_{k=1}^{\infty} \frac{c}{k(k+1)} = c \sum_{k=1}^{\infty} \frac{1}{k(k+1)}$$

$$= c \left( 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots \right) = c$$

$$E(X) = \sum_{k=1}^{\infty} k \cdot P_X(k) = \sum_{k=1}^{\infty} k \cdot \frac{1}{k(k+1)} = \infty.$$
Expectation of Continuous Random Variable (with density)

\(X\) discrete, \(X \in \{t_1, t_2, t_3, \ldots\}\)

\[E(X) = \sum_j t_j \Pr(X = t_j)\]

\[\sum_t t \cdot p_X(t)\]

\(X\) continuous (\(\Pr(X = t) = 0\) for each \(t \in \mathbb{R}\))

\[E(X) = \int_{-\infty}^{\infty} t f_X(t) \, dt\]

Eg. Let \(U \sim \text{Unif}([a, b])\), \(f_U(t) = \begin{cases} \frac{1}{b-a} & a \leq t \leq b \\ 0 & \text{otherwise} \end{cases}\)

\[E(U) = \int_{-\infty}^{\infty} t f_U(t) \, dt = \int_a^b t \cdot \frac{1}{b-a} \, dt = \frac{1}{b-a} \cdot \frac{1}{2} t^2 \bigg|_{t=a}^{t=b} = \frac{1}{b-a} \left( \frac{b^2-a^2}{2} \right) = \frac{a+b}{2}\]
Question:
Shoot an arrow at a circular target of radius 1. What is the expected distance of the arrow from the center?

(a) 1
(b) 2/3
(c) 1/2
(d) 1/4
(e) 0

\[ X = \text{dist. from center} \]
\[
F_x(r) = \begin{cases} 
0 & r \leq 0 \\
1 & 0 < r \leq 1 \\
1 & r > 1 
\end{cases}
\]

\[
f_x(r) = \begin{cases} 
0 & r < 0 \\
2r & 0 \leq r \leq 1 \\
0 & r > 1 
\end{cases}
\]

\[
\mathbb{E}(X) = \int_{0}^{\infty} r f_x(r) \, dr
\]
\[
= \int_{0}^{1} r \cdot 2r \, dr = \frac{2}{3} r^3 \bigg|_0^1 = \frac{2}{3}
\]

\[
\left( \frac{\text{Area (IDz^1/3)}}{\text{Area (ID)}} \right) = \frac{\pi (2/3)^2}{\pi 1^2} = \frac{4}{9}
\]
E.g. \( f(t) = \begin{cases} 0 & t \leq 1 \\ \frac{1}{t^2} & t > 1 \end{cases} \) is a probability density.

\[
\int_{-\infty}^{\infty} f(t) \, dt = \int_{1}^{\infty} \frac{1}{t^2} \, dt = -\frac{1}{t} \bigg|_{1}^{\infty} = \left( \frac{1}{\infty} - \frac{1}{1} \right) = 1
\]

If \( X \) is a continuous random variable with \( f_X = f \), what is \( \mathbb{E}(X) \)?

\[
\mathbb{E}(X) = \int_{-\infty}^{\infty} t \cdot f(t) \, dt = \int_{1}^{\infty} t \cdot \frac{1}{t^2} \, dt = \int_{1}^{\infty} \frac{1}{t} \, dt
\]

\[
= \ln |t| \bigg|_{1}^{\infty} = \ln 1 - \ln 1 = 0
\]

\[
= \infty
\]
Expectations of Functions of Random Variables

\[ \Omega \rightarrow \mathbb{R} \rightarrow \mathbb{R} \]
random variable \( x \)
function \( g \)

\( g(x) = g \circ x \) is a new random variable.

E.g. \( X \sim \text{Bin}(n, p) \) is the number of successes in \( n \) trials
\( g(x) = x/n \) is the proportion of successful trials.

\[ g(x) \epsilon \{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n}{n} = 1\} \]

\[ E(g(X)) = \sum_{t} t \cdot P(g(X) = t) \]
\[ = \sum_{k=0}^{n} \frac{k}{n} \cdot P\left(\frac{x}{n} = \frac{k}{n}\right) \]
\[ = \sum_{k=0}^{n} \frac{k}{n} \cdot P(X = k) \]

In general: for a discrete random variable

**Proposition:** \( E(g(X)) = \sum_{s} g(s) \cdot P(X = s) \).

Note, by definition, \( E(g(X)) = \sum_{t} t \cdot P(g(X) = t) \).
**Proposition:** for a discrete random variable

\[ E(g(X)) = \sum_s g(s) P(X=s). \]

\[ \text{Pf. We know} \quad \sum_t \sum_{s: g(s)=t} P(X=s) \]

\[ = \sum_t \sum_{s: g(s)=t} P(X=s) \]

\[ = \sum_t \sum_{s: g(s)=t} g(s) P(X=s) \]
**Proposition:** For a continuous random variable $X$ with probability density $f_X$, 

$$E(g(X)) = \int_{-\infty}^{\infty} g(t) f_X(t) \, dt.$$ 

**Eg.** Let $U$ be a uniform random variable on $[a, b]$. Then 

$$E(U^2) = \int_{a}^{b} t^2 f_U(t) \, dt = \int_{a}^{b} \frac{1}{b-a} \cdot t^2 \, dt = \frac{1}{b-a} \cdot \frac{1}{3} b^3 \left|_{a}^{b} \right.$$ 

$$f_U(t) = \begin{cases} 
\frac{1}{b-a} & a \leq t \leq b \\
0 & \text{otherwise}
\end{cases}$$ 

$$= \frac{b^3-a^3}{3(b-a)} = \frac{2^2+a^2+b^2}{3}.$$ 

($a=0, b=1, \ E(U_{[0,1]}^2) = \frac{1}{3}.\ )$
Eg. Recall the car accident insurance example. An accident causes $Y$ of damage to your car, where your insurance deductible is $500. What is the expected amount you pay?

$Y \sim \text{Unif}([100, 1500])$.

$X = \text{amount you pay}$

(neither continuous nor discrete r.v.)

$X = \min(Y, 500) = g(Y)$

$E(X) = E(g(Y)) = \int_{100}^{1500} \min(t, 500) \frac{1}{1400} \, dt$

$E(X) = \int_{100}^{500} t \cdot \frac{1}{1400} \, dt + \int_{500}^{1500} 500 \cdot \frac{1}{1400} \, dt = \frac{3100}{7} + 442.86 = 442.86$. 