Math 180A: Introduction to Probability

Lecture B00 (Nemish)

math.ucsd.edu/~ynemish/teaching/180a

Lecture C00 (Au)

math.ucsd.edu/~bau/f20.180a

Today: ASV 3.4 (Variance)

ASV 3.5 (Gaussian distribution)

Video: Prof. Todd Kemp, Fall 2019

Next: ASV 3.5, 4.1

Week 5: Quiz 3 (Wednesday, Nov 4)

Homework 4 (due Friday, Nov 6)

Regrades for Homework 3 (Nov 2-3)

Variance

Definition: The variance of a random variable X is

$$\mathrm{Var}(X) = \mathbb{E}\left[(X-\mathbb{E}(X))^2\right].$$

I.e. first compute $\mu := \mathbb{E}(X)$. Then apply the function $g(x) = (x-\mu)^2$ to X, and compute $\mathbb{E}(g(X))$.

* If X is discrete, $\mathrm{Var}(X) = \sum_t (t-\mu)^2\, \mathbb{P}(X=t)$

* If X is continuous, $\mathrm{Var}(X) = \int_{-\infty}^{\infty} (t-\mu)^2 \underline{\underline{f_X(t)}}\, dt$

In any case, $\mathrm{Var}(X) \geq 0$. Its square root is

$\sigma(X) = \sqrt{\mathrm{Var}(X)}$ is called standard deviation.

Eg. If $X \sim \mathrm{Ber}(p)$, $\mathbb{E}(X) = p$, $\therefore$

$$\mathrm{Var}(X) = \mathbb{E}((X-p)^2) = \sum_t (t-p)^2 \, \mathbb{P}(X=t)$$

$$= (1-p)^2 \, \mathbb{P}(X=1) + (0-p)^2 \, \mathbb{P}(X=0)$$

$$= (1-p)^2 p + p^2 (1-p)$$

$$= (1-p)p\,[1-p+p]$$

$$= \boxed{(1-p)p}$$

$p = \frac{1}{2}$

$\mathrm{Var} = \frac{1}{4}$

$\sigma = \frac{1}{2}$

Eg. If $U \sim \mathrm{Unif}([a,b])$, $\mathbb{E}(U) = \frac{a+b}{2}$, $\therefore$

$$\mathrm{Var}(U) = \int_{-\infty}^{\infty} \left(t - \frac{a+b}{2}\right)^2 f_U(t)\,dt = \int_a^b \left(t - \frac{a+b}{2}\right)^2 \cdot \frac{1}{b-a}\,dt$$

$$f_U(t) = \begin{cases} \frac{1}{b-a} & \text{on } [a,b] \\ 0 & \text{otherwise} \end{cases}$$

$$= \frac{1}{3}\left(t - \frac{a+b}{2}\right)^3 \Bigg|_{t=a}^{t=b} \cdot \frac{1}{b-a}$$

$$= \frac{1}{3} \cdot \frac{1}{b-a} \cdot \left[\left(b - \frac{a+b}{2}\right)^3 - \left(a - \frac{a+b}{2}\right)^3\right]$$

$$= \frac{1}{3} \cdot \frac{1}{b-a} \left[\left(\frac{b-a}{2}\right)^3 - \left(\frac{a-b}{2}\right)^3\right]$$

$$= \frac{1}{3 \cdot 2^3} \cdot \frac{1}{b-a} \cdot 2(b-a)^3$$

$$\frac{(b-a)^3}{12(b-a)} = \frac{(b-a)^2}{12}$$

$$\text{St. Dev.} = \frac{|b-a|}{\sqrt{12}}$$

Variance is a measure of how "spread out from the mean" the distribution is. For example:

Theorem: Let X be a random variable with finite expectation $\mathbb{E}(X)=\mu$. Then

$$\operatorname{Var}(X)=0 \quad \text{iff} \quad \mathbb{P}(X=\mu)=1.$$

(i.e. $X \equiv \mu$.)

Pf. ($\Longleftarrow$) If $\mathbb{P}(X=\mu)=1$. So X is discrete.

$\therefore \operatorname{Var}(X) = \mathbb{E}((X-\mu)^2) = (\mu-\mu)^2\,\mathbb{P}(X=\mu) = 0.$

($\Longrightarrow$) [Here we will assume X is discrete, for now.]

$$0 = \operatorname{Var}(X) = \underbrace{\sum_t \underbrace{(t-\mu)^2}_{\geq 0}\,\underbrace{\mathbb{P}(X=t)}_{\geq 0}}_{\text{sum of } \geq 0}$$

$\Rightarrow$ for each t, $(t-\mu)^2 \cdot \mathbb{P}(X=t)=0$.

$\therefore$ For each t, either $\underline{(t-\mu)^2=0}$ or $\mathbb{P}(X=t)=0$.

If $t \neq \mu \Longrightarrow \mathbb{P}(X=t)=0$.

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Chebyshev's Inequality

If X has finite mean $\mathbb{E}(X)=\mu$ and finite variance $Var(X)=\sigma^2$, then

$$\mathbb{P}(|X-\mu|\geq k\sigma) \leq \frac{1}{k^2}$$

Two ideas in the proof:

(1) Indicators. $\mathbb{1}_A = \begin{cases} 1 & \text{if } A \text{ happens} \\ 0 & \text{if } A \text{ does not} \end{cases}$

$\mathbb{1}_A$ is a 0-1 valued r.v. It is a $Ber(p)$ with $p=\mathbb{P}(A)$

$$\therefore \mathbb{E}(\mathbb{1}_A) = \mathbb{P}(A)$$

(2) Monotonicity. FACT: if $X \leq Y$ then $\mathbb{E}(X) \leq \mathbb{E}(Y)$

(Intuitive; for proof, we must wait until week 8.)

Proof of Chebyshev:

$$\mathbb{P}(|X-\mu|\geq k\sigma) = \mathbb{E}(\mathbb{1}_{\{|X-\mu|\geq k\sigma\}})$$
$$= \mathbb{E}\left(\mathbb{1}_{\left(\frac{X-\mu}{k\sigma}\right)^2\geq 1}\right)$$
$$\leq \mathbb{E}\left[\left(\frac{X-\mu}{k\sigma}\right)^2\right] = \frac{1}{k^2\cancel{\sigma^2}}\frac{\mathbb{E}[(X-\mu)^2]}{\cancel{\sigma^2}}$$

$$|X-\mu|\geq k\sigma \iff \underbrace{\left(\frac{X-\mu}{k\sigma}\right)^2}_{Y} \geq 1$$

if $Y\geq 1$ then $Y \geq \mathbb{1}_{\{Y\geq 1\}}$

Normal (Gaussian) Distribution

3.5

The standard normal distribution $\mathcal{N}(0,1)$ is given by the density

$$f(t) = \underbrace{\frac{1}{\sqrt{2\pi}}\, e^{-t^2/2}}_{>0}$$

Check:

$$I = \int_{-\infty}^{\infty} e^{-t^2/2}\, dt = \sqrt{2\pi}$$

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2/2}\,dx \int_{-\infty}^{\infty} e^{-y^2/2}\,dy$$

$$= \iint_{\mathbb{R}^2} \underbrace{e^{-x^2/2} e^{-y^2/2}}_{e^{-(x^2+y^2)/2}}\, dx\,dy$$

$$= \int_0^{2\pi} d\theta \int_0^{\infty} \underbrace{r\,dr\, e^{-r^2/2}}$$

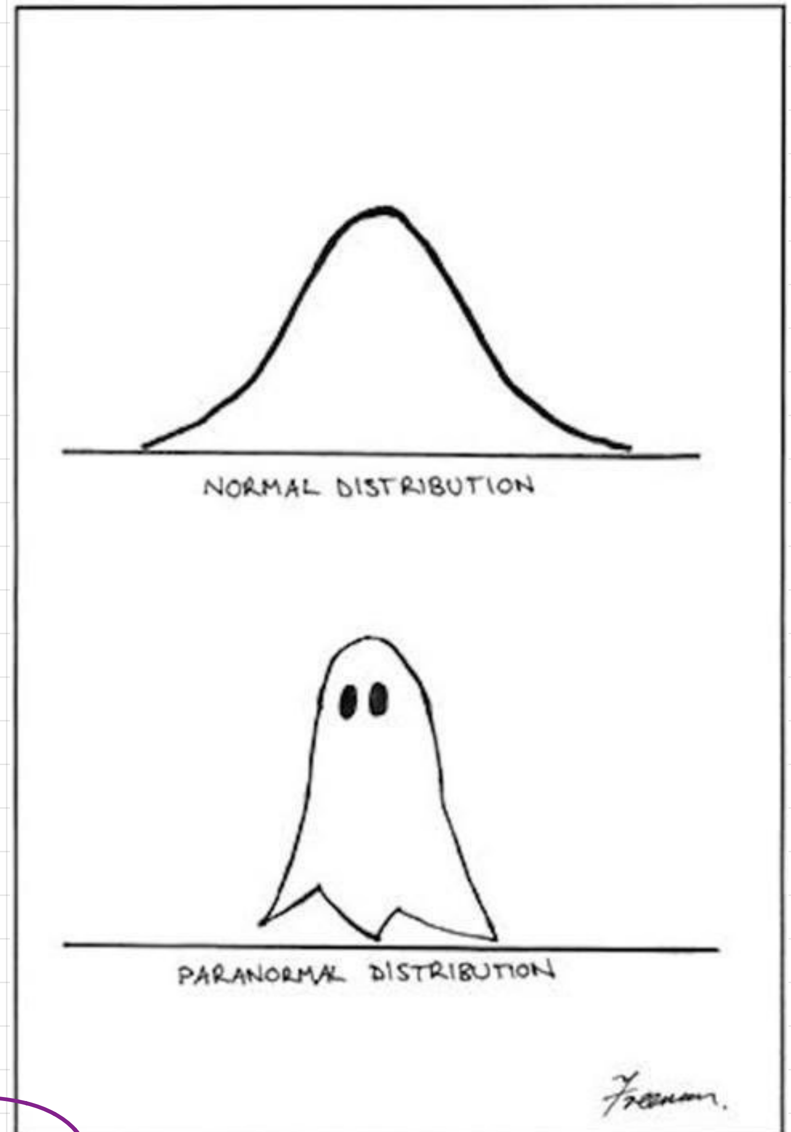
$$= 2\pi \cdot \left(-e^{-r^2/2}\right)\Big|_0^{\infty} = 2\pi \cdot 1$$

Polar Coords:

$(x,y) \to (r,\theta)$

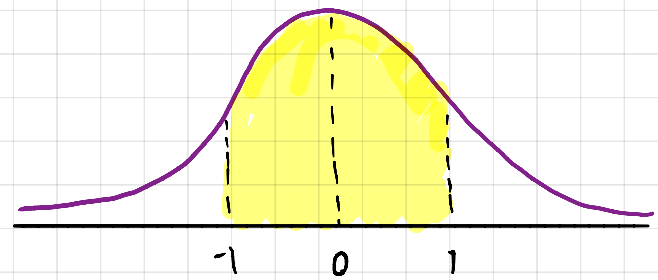
$dx\,dy \to r\,dr\,d\theta$

$x^2+y^2 = r^2$



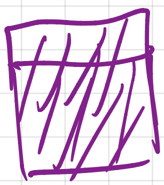
CDF of $\mathcal{N}(0,1)$

Suppose $X \sim \mathcal{N}(0,1)$. What is $\mathbb{P}(|X| \le 1)$?



$\mathbb{P}(-1 \le X \le 1)$

$$= \int_{-1}^{1} f_X(t)\,dt = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-t^2/2}\,dt$$

$\int_{-1}^{1}\int_{-1}^{1} \cdots$  ← not good for polar coords.

$$\Phi(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt$$