Math 180A: Introduction to Probability

Lecture B00 (Nemish)

math.ucsd.edu/~ynemish/teaching/180a

Lecture C00 (Au)

math.ucsd.edu/~bau/f20.180a

Today: ASV 5.1 (Moment generating function)

Video: Prof. Todd Kemp, Fall 2019

Next: ASV 5.2, 6.1

Week 5: Quiz 4 (Wednesday, Nov 18 on Lectures 11-14)

Homework 6 (due Friday, Nov 20)

Regrades for Homework 4 (Nov 16-17)

Midterm 2 (Monday, Nov 23)

Functions to Describe Probability Distributions

5.1

Random variable X.

* CDF $F_X(t) = \mathbb{P}(X \leq t)$
  - Works every time
  - But can be hard to compute.
  - No clear relation to $\mathbb{E}(X)$

* PMF $p_X(k) = \mathbb{P}(X = k)$
  - Only when X is discrete.
  - $\mathbb{E}(X) = \sum_k k\, p_X(k)$

* PDF $f_X(t) = \frac{d}{dt} F_X(t)$
  - Only when X is continuous.
  - $\mathbb{E}(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$

New entry: MGF Moment Generating Function.

$$M_X(t) = \mathbb{E}(e^{tX}) \begin{cases} = \sum_k e^{tk} p_X(k) & \text{if } X \text{ discrete} \\ = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx & \text{if } X \text{ contin.} \end{cases}$$

Eg. $X \sim \text{Ber}(p)$ $\quad \mathbb{P}(X=1)=p$, $\mathbb{P}(X=0)=1-p$

$$\therefore M_X(t) = \mathbb{E}(e^{tX}) = e^{t\cdot 0}(1-p) + e^{t\cdot 1}(p)$$

$$p(e^t-1)+1 \leftarrow = e^t p + (1-p)$$

Eg. $N \sim \text{Poisson}(\lambda)$ $\quad \mathbb{P}(N=k) = e^{-\lambda}\frac{\lambda^k}{k!}$

$$M_N(t) = \mathbb{E}(e^{tN}) = \sum_{k=0}^{\infty} e^{tk} e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{(e^t\lambda)^k}{k!} = e^{-\lambda}e^{e^t\lambda}$$

$$(e^t)^k\lambda^k = (e^t\lambda)^k$$

$$= e^{e^t\lambda-\lambda}$$

$$= e^{\lambda(e^t-1)}$$

Eg. $Z \sim \mathcal{N}(0,1)$

$$M_Z(t) = \mathbb{E}(e^{tZ}) = \int_{-\infty}^{\infty} e^{tx}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}+tx}\,dx$$

$$\left[-\frac{x^2}{2}+tx = -\frac{1}{2}(x^2-2tx) = -\frac{1}{2}(x^2-2tx+t^2-t^2) = -\frac{1}{2}(\underbrace{x^2-2tx+t^2}_{(x-t)^2}) + t^2/2 \right]$$

$$M_Z(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \underbrace{e^{-\frac{1}{2}(x-t)^2+\frac{t^2}{2}}}_{e^{-\frac{1}{2}(x-t)^2}\cdot e^{t^2/2}}\,dx = e^{\frac{t^2}{2}}\cdot\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x-t)^2/2}\,dx}_{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-u^2/2}\,du = 1} = e^{t^2/2}$$

E.g. $T \sim \text{Exp}(\lambda)$

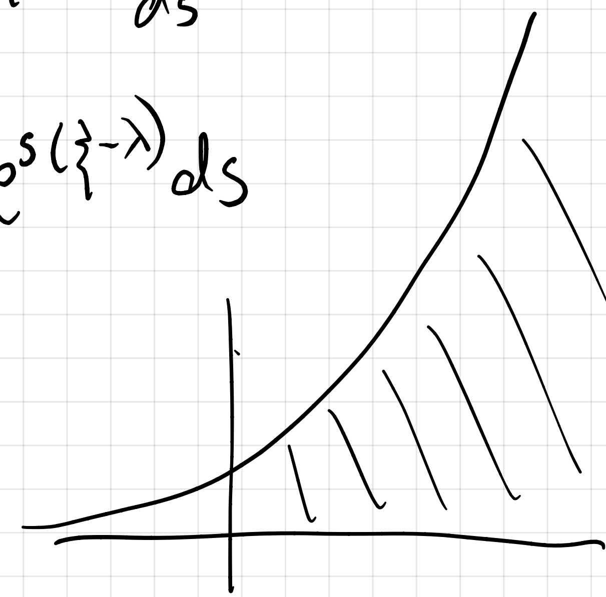
$$f_T(s) = \begin{cases} \lambda e^{-\lambda s} & s > 0 \\ 0 & s \le 0 \end{cases}$$

$$M_T(\xi) = \mathbb{E}(e^{\xi T}) = \int_0^\infty e^{\xi s} \cdot \lambda e^{-\lambda s}\, ds$$

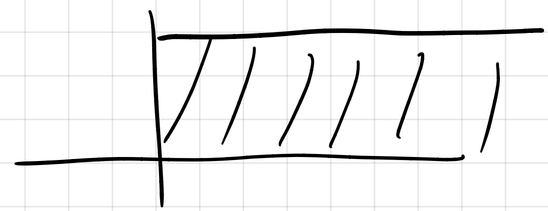
$$= \lambda \int_0^\infty e^{\xi s - \lambda s}\, ds$$

$$= \lambda \int_0^\infty e^{s(\xi - \lambda)}\, ds$$

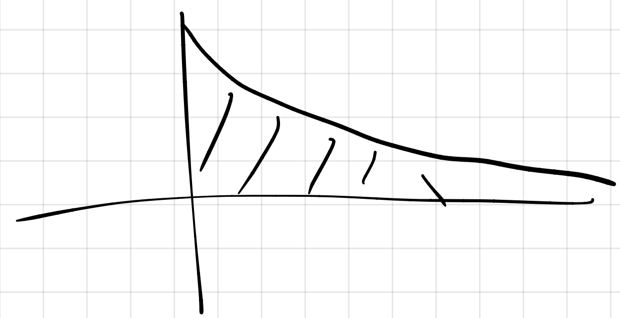
If $\xi - \lambda > 0$



If $\xi - \lambda = 0$



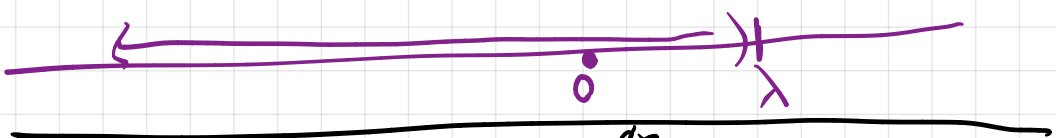
If $\xi - \lambda < 0$



$$M_T(\xi) = \infty \quad \text{if } \xi \ge \lambda$$

$$= \lambda \cdot \frac{-1}{b}$$

$$= \frac{\lambda}{\lambda - \xi} \quad \text{if } \xi < \lambda$$



If $b = \xi - \lambda < 0$

$$\int_0^\infty e^{bs}\, ds = \frac{1}{b} e^{bs} \Big|_0^\infty = \frac{1}{b}(0 - 1)$$

A MGF may take some infinite values.
There is always at least one finite value:

$$M_X(0) = \mathbb{E}(e^{0 \cdot X}) = \mathbb{E}(1) = 1.$$

But it can happen that there are no others!

Eg. Cauchy density $f(x) = \dfrac{1}{\pi(1+x^2)} \sim X$

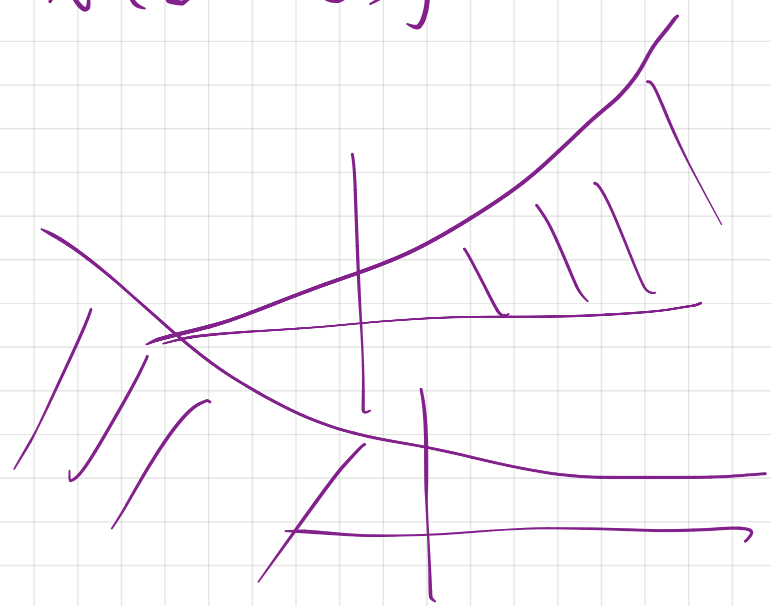
$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \frac{1}{\pi}\tan^{-1}(x)\Big|_{-\infty}^{\infty} = \frac{1}{\pi}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 1.$$

$$\mathbb{E}(e^{tX}) = \frac{1}{\pi}\int \underbrace{e^{tx} \cdot \frac{1}{1+x^2}\,dx}$$

If $t>0$, $\dfrac{e^{tx}}{1+x^2}$

If $t<0$, $\dfrac{e^{tx}}{1+x^2}$



Why $\underline{M}$GF?

Given a random variable X, its <u>moments</u> (should they exist) are the numbers $\mathbb{E}(X^k)$, $k=0,1,2,\ldots$

These can be computed from $M_X(t)$ as follows.

$$\frac{d}{dt}\mathbb{E}(e^{tX}) \underset{\text{faith}}{\uparrow}{=} \mathbb{E}\left(\frac{d}{dt}e^{tX}\right) = \mathbb{E}(Xe^{tX})$$

faith

$$\therefore \left(\frac{d}{dt}\mathbb{E}(e^{tX})\right)\Big|_{t=0} = \mathbb{E}(Xe^{0\cdot X}) = \mathbb{E}(X) \quad \leftarrow \text{mean}$$

$$\vdots$$

$$\frac{d^k}{dt^k}M_X(0) = \mathbb{E}(X^k).$$

Theorem: Suppose $M_X(t) < \infty$ for all t in some neighborhood of 0

$(-\varepsilon, \varepsilon)$ $\varepsilon > 0$.

Then M_X is analytic on this neighborhood: its Taylor series based @ 0 converges to $M_X(t)$ on this interval, and

$$M_X(t) = \sum_{k=0}^{\infty} \mathbb{E}(X^k) \frac{t^k}{k!}$$

Eg. Find the moments of the $\text{Exp}(\lambda)$ distribution.

$$M_T(t) = \frac{\lambda}{\lambda - t} \qquad t < \lambda$$

0 λ

$$= \frac{1}{1 - t/\lambda} = \sum_{k=0}^{\infty} \left(\frac{t}{\lambda}\right)^k = \sum_k \frac{1}{\lambda^k} \cdot t^k$$

$\left|\frac{t}{\lambda}\right| < 1$

$$\therefore \frac{\mathbb{E}(T^k)}{k!} = \frac{1}{\lambda^k}$$

$$\mathbb{E}(T^k) = \frac{k!}{\lambda^k}.$$

$$\mathbb{E}(T) = \frac{1}{\lambda} \qquad \mathbb{E}(T^2) = \frac{2}{\lambda^2} \qquad \text{Var}\, T = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

Eg. Find the moments of the $\mathcal{N}(0,1)$ distribution.

$$M_Z(t) = e^{t^2/2} = \sum_{k=0}^{\infty} \frac{(t^2/2)^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{2^k k!} t^{2k}$$

$$= \sum_{n=0}^{\infty} E(Z^n)\frac{t^n}{n!} = \sum_{n=2k,\ k=0}^{\infty} E(X^{2k})\frac{t^{2k}}{(2k)!}$$

$E(Z^n) = 0$ if n is odd

$$\therefore \frac{E(X^{2k})}{(2k)!} = \frac{1}{2^k k!} \qquad \therefore E(X^{2k}) = \frac{(2k)!}{2^k k!}$$

$$= \frac{\cancel{2}k(2k-1)(\cancel{2}k-2)(2k-3)(\cancel{2}k-4)\cdots(3)\cancel{(2)}(1)}{\cancel{2}k\ \ \cancel{2}(k-1)\ \ \cancel{2}(k-2)\ \ \cancel{2}(1)}$$

pairings of $2k$ things $\rightarrow$

$$= (2k-1)(2k-3)(2k-5)\cdots(3)(1)$$

$$= (2k-1)!!$$