MATH 180A: Introduction to Probability

Lecture B00 (Nemish)

www.math.ucsd.edu/~ynemish/teaching/180a

Lecture C00 (Au)

www.math.ucsd.edu/~bau/f20.180a

Today: Joint distributions. Independence of random variables

Video: Prof. Todd Kemp, Fall 2019

Next: ASV 8.1-8.3

Week 9:

- Homework 7 (due Friday, December 4, 11:59 PM)
- Regrades for Homework 6 (until Wednesday, December 2, 11 PM)

Jointly Continuous Random Vectors

A random vector $\underline{X} = (X_1, \dots, X_n)$ has a pdf $f_{\underline{X}} : \mathbb{R}^n \to \mathbb{R}_+$ if, for "nice" subsets $B \subseteq \mathbb{R}^n$

$$\mathbb{P}(\underline{X} \in B) = \int_B f_{\underline{X}}(x_1, \dots, x_n)\, dx_1 \cdots dx_n .$$

(we say $X_1, X_2, \dots, X_n$ are jointly continuous.)

Properties: $f_X(\underline{x}) \geq 0$ for all $\underline{x} \in \mathbb{R}^n$

$$\int_{\mathbb{R}^n} f(\underline{x})\, d\underline{x} = 1 .$$

$\|\underline{x}\|^2 = x_1^2 + \cdots + x_n^2$

Eg. Standard Multivariate Normal $f(\underline{x}) = (2\pi)^{-\frac{n}{2}} e^{-\|\underline{x}\|^2/2}$

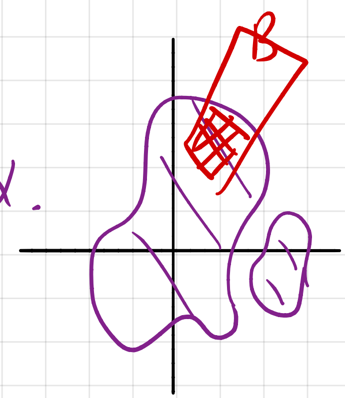
$(n=2)$ $f(x,y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2} = \frac{1}{2\pi} e^{-x^2/2} e^{-y^2/2} > 0$

$$\therefore \iint_{\mathbb{R}^2} f(x,y)\, dx\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-x^2/2} e^{-y^2/2}\, dx\, dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-y^2/2} \Big(\underbrace{\int_{-\infty}^{\infty} e^{-x^2/2}\, dx}_{\sqrt{2\pi}}\Big) dy = \frac{(\sqrt{2\pi})^2}{2\pi} = 1 .$$

$\sqrt{2\pi}$

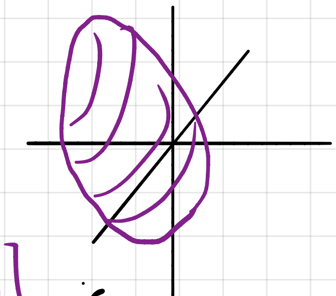
E.g. Uniform Probability

Let A be a bounded region in $\mathbb{R}^2$ with area α.



$$\mathbb{P}((X,Y)\in B) = \frac{\text{Area}(B\cap A)}{\text{Area}(A)}$$

Let V be a bounded region in $\mathbb{R}^3$ with volume ϑ



$\vdots$

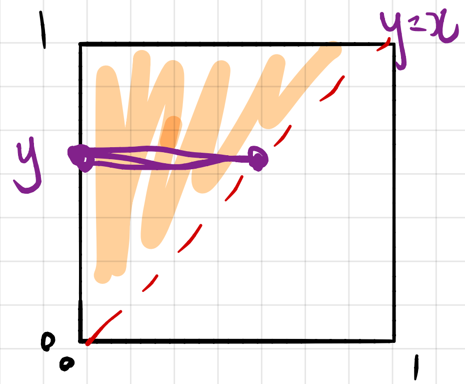
A random vector $\underline{X} = (X_1, X_2)$ [resp. $\underline{X} = (X_1, X_2, X_3)$] is uniformly distributed in A [resp. V] if it is jointly continuous and has density

$$f_A(x_1,x_2) = \frac{1}{\alpha}\mathbb{1}((x_1,x_2)\in A) \quad \text{resp. } f_V(x_1,x_2,x_3) = \frac{1}{\vartheta}\mathbb{1}((x_1,x_2,x_3)\in V)$$

$$\begin{cases} =1 & \text{if } (x_1,x_2)\in A \\ =0 & \text{if } (x_1,x_2)\notin A \end{cases}$$

$$\mathbb{P}((X,Y)\in B) := \iint_B f_A(x_1,x_2)\,dx_1 dx_2 = \iint_{\underline{B}} \frac{1}{\alpha}\underline{\mathbb{1}\{(x_1,x_2)\in A\}}\,dx_1 dx_2 = \iint_{A\cap B} \frac{1}{\alpha}\,dx_1 dx_2 = \text{Area}(A\cap B)/\alpha.$$

Eg. Suppose (X, Y) has joint density $f(x,y) = \frac{3}{2}(xy^2+y)\,\mathbb{1}(0 \leq x,y \leq 1)$



$$\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy = \int_0^1\int_0^1 \frac{3}{2}(xy^2+y)\,dx\,dy$$

$$= \frac{3}{2}\int_0^1 dy \left(y^2 \cdot \frac{x^2}{2} + yx\right)\Big|_{x=0}^{1} = \frac{3}{2}\int_0^1 \left(\frac{1}{2}y^2+y\right)dy$$

$$= \frac{3}{2}\left(\frac{1}{6}y^3 + \frac{1}{2}y^2\right)\Big|_{y=0}^{y=1}$$

$$= \frac{3}{2}\left(\frac{1}{6}+\frac{1}{2}\right) = 1.$$

Compute $\mathbb{P}(X < Y)$.

$T = \{(x,y) : x < y\}$

$$= \mathbb{P}((X,Y) \in T) = \iint_T \frac{3}{2}(xy^2+y)\,dx\,dy$$

$$= \frac{3}{2}\int_0^1 dy \int_0^y dx\,(xy^2+y) = \frac{3}{2}\int_0^1 dy\left(y^2 \cdot \frac{x^2}{2} + yx\right)\Big|_{x=0}^{y}$$

$$= \frac{3}{2}\int_0^1 dy\left(y^2 \cdot \frac{y^2}{2} + y^2\right)$$

$$= \frac{3}{2}\left(\frac{1}{10}y^5 + \frac{1}{3}y^3\right)\Big|_0^1 = \frac{3}{2}\left(\frac{1}{10}+\frac{1}{3}\right)$$

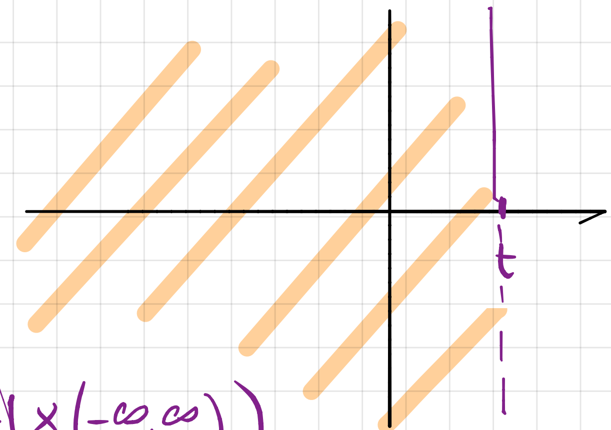
$$= 13/20.$$

Marginals

Let $\underline{X} = (X_1, X_2, \ldots, X_n)$ be a jointly continuous random vector, with joint density $f_{\underline{X}}$. The density of X_j is

$$f_{X_j}(t) = \int_{\mathbb{R}^{n-1}} f_{\underline{X}}(x_1, x_2, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_n)\, dx_1 \cdots dx_{j-1}\, dx_{j+1} \cdots dx_n .$$

(Integrate out all but the j^{th} variable)



Proof $(n=2, j=1)$ $\underline{X} = (X, Y)$

$$\mathbb{P}(X \le t) = \mathbb{P}(X \le t, Y \in \mathbb{R}) = \mathbb{P}((X,Y) \in (-\infty, t] \times (-\infty, \infty))$$

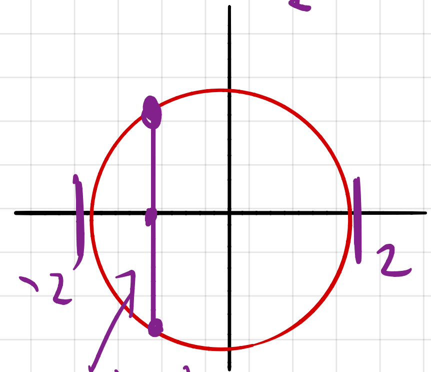
$$F_X(t) = \int_{-\infty}^{t} dx \int_{-\infty}^{\infty} dy\, f_{(X,Y)}(x,y) = \int_{-\infty}^{t} g(x)\,dx \qquad g(x) = \int_{-\infty}^{\infty} f_{(X,Y)}(x,y)\,dy$$

$$\therefore f_X(t) = \frac{d}{dt} F_X(t) = \frac{d}{dt} \int_{-\infty}^{t} g(x)\,dx = g(t)$$

FTC

E.g. Suppose $\underline{X}$ is uniformly distributed on the disk of radius 2.
Find the marginal density of X_1 $\quad (\underline{X}=(X_1,X_2))$

$t^2+x_2^2=4$



$$f_{\underline{X}}(x_1,x_2) = \frac{1}{4\pi}\mathbb{1}_{\mathbb{D}_2}(x_1,x_2)$$

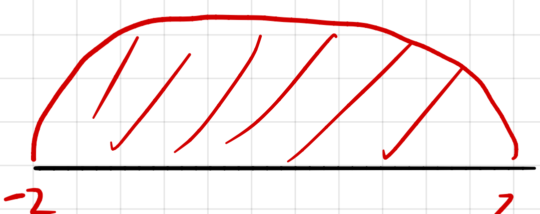
$$f_{X_1}(t) = \int_{-\infty}^{\infty} f_{\underline{X}}(t,x_2)\,dx_2 = \frac{1}{4\pi}\int_{-\infty}^{\infty} \underbrace{\mathbb{1}_{\mathbb{D}_2}(t,x_2)}\,dx_2$$

Constant, $-\sqrt{4-t^2}\le x_2\le\sqrt{4-t^2}$

$-2\le t\le 2$

$$= \frac{1}{4\pi}\int_{-\sqrt{4-t^2}}^{\sqrt{4-t^2}} dx_2 = \frac{1}{4\pi}\left(2\sqrt{4-t^2}\right)$$

$$= \begin{cases} \dfrac{\sqrt{4-t^2}}{2\pi} & -2\le t\le 2 \\ 0 & |t|>2 \end{cases}$$



"Semicircle Law"

Expectation

If $\underline{X}$ is a random vector in $\mathbb{R}^n$ with joint density $f_{\underline{X}} : \mathbb{R}^n \to \mathbb{R}_+$, and $g : \mathbb{R}^n \to \mathbb{R}$, then

$$\mathbb{E}(g(\underline{X})) = \int_{\mathbb{R}^n} g(\underline{x}) f_{\underline{X}}(\underline{x}) \, d\underline{x}$$

Eg. $\underline{X} = (X, Y)$, $f_{\underline{X}}(x,y) = \frac{3}{2}(xy^2 + y) \, \mathbb{1}(0 \leq x, y \leq 1)$. Find $\mathbb{E}(X^2 Y)$.

$$\mathbb{E}(X^2 Y) = \iint_{\mathbb{R}^2} x^2 y \, f_{\underline{X}}(x,y) \, dx\,dy = \int_0^1 \int_0^1 x^2 y \cdot \frac{3}{2}(xy^2 + y) \, dx\,dy$$

Exercise

$$= \frac{25}{96}.$$

CAUTION !

If $X_1, X_2, \ldots, X_n$ are discrete random variables, then $\underline{X} = (X_1, \ldots, X_n)$ is a (jointly) discrete random vector.

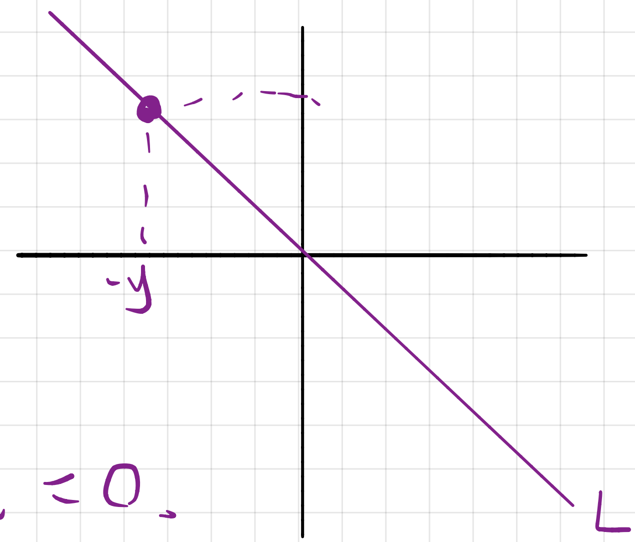
BUT

Just because $X_1, \ldots, X_n$ are (separately) continuous random variables does <u>not</u> necessarily imply that $\underline{X} = (X_1, \ldots, X_n)$ has a joint density!

E.g. $X \sim N(0,1)$, $Y = -X \sim N(0,1)$

$$\mathbb{P}(Y \leq t) = \mathbb{P}(-X \leq t) = \mathbb{P}(X \geq -t) = 1 - \mathbb{P}(X < -t) = 1 - \Phi(-t) = \Phi(t)$$

Must have $\mathbb{P}(X = -Y) = 1$.



If $f_{(X,Y)}$ existed,

$$\mathbb{P}(X = -Y) = \iint_L f_{(X,Y)}(x,y)\,dx\,dy$$

$$\int_{-\infty}^{\infty} dy \underbrace{\int_{-y}^{-y} f_{(X,Y)}(x,y)\,dx}_{=0} = 0.$$

Joint Distributions & Independence

6.3

Suppose $\underline{X} = (X_1, \dots, X_n)$ is jointly continuous. Then $X_1, \dots, X_n$ are independent iff $f_{\underline{X}}(x_1, \dots, x_n) = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_n}(x_n)$.

$(n=2:$ $\quad \underline{f_{(X,Y)}(x,y) = f_X(x) f_Y(y)}$.

Pf. $(\Leftarrow)$ $\quad \mathbb{P}(X \in A, Y \in B) = \mathbb{P}((X,Y) \in A \times B)$

$$= \iint_{A \times B} f_X(x) f_Y(y) \, dx \, dy$$

$$= \int_A f_X(x) \, dx \int_B f_Y(y) \, dy = \mathbb{P}(X \in A) \mathbb{P}(Y \in B)$$

$\therefore$ X, Y independent.

$(\Rightarrow)$ NEXT TIME.