MATH 180A: Introduction to Probability

Lecture B00 (Nemish)

www.math.ucsd.edu/~ynemish/teaching/180a

Lecture C00 (Au)

www.math.ucsd.edu/~bau/f20.180a

Today: Independence of random variables. Convolutions

Next: ASV 8.2-8.3

Video: Prof. Todd Kemp, Fall 2019

Week 9:

- Homework 7 (due Friday, December 4, 11:59 PM)
- Regrades for Homework 6 (until Wednesday, December 2, 11 PM)

Joint Distributions & Independence

6.3

Suppose $\underline{X} = (X_1, \ldots, X_n)$ is jointly continuous. Then $X_1, \ldots, X_n$ are independent iff $f_{\underline{X}}(x_1, \ldots, x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n)$.

($\Longrightarrow$) (Special case $n=2$)

Suppose X, Y are independent. f_X, f_Y.

$\Downarrow$

$$\mathbb{P}(X \le s, Y \le t) = \mathbb{P}(X \le s)\,\mathbb{P}(Y \le t)$$

$$\mathbb{P}\big((X,Y) \in (-\infty, s] \times (-\infty, t]\big) \qquad \int_{-\infty}^{s} f_X(x)\,dx \cdot \int_{-\infty}^{t} f_Y(y)\,dy$$

$$\int_{-\infty}^{t}\int_{-\infty}^{s} f_{X,Y}(x,y)\,dx\,dy \quad \xleftarrow{\text{equal}} \text{ for all } s, t. \quad \int_{-\infty}^{t}\int_{-\infty}^{s} f_X(x)\, f_Y(y)\,dx\,dy$$

Now take $\frac{\partial^2}{\partial s \partial t}$ of both sides.

$$f_{X,Y}(s,t) = f_X(s)\, f_Y(t).$$

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Eg. Multivariate standard normal $\frac{1}{(2\pi)^{n/2}} e^{-\|\underline{x}\|^2/2}$

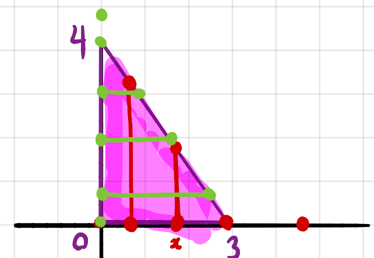
$$= \left(\frac{1}{\sqrt{2\pi}} e^{-x_1^2/2}\right) \cdots \left(\frac{1}{\sqrt{2\pi}} e^{-x_n^2/2}\right)$$

Since the joint density is a product of single-variable densities, the components of the random vector are independent, with those densities as marginals:

I.e. if $\underline{X}$ is a multivariate standard normal, $\underline{X} = (X_1, \dots, X_n)$ then $X_1, \dots, X_n$ are independent $N(0,1)$'s.

Q: S'pose $\underline{X}$ is uniform on the triangle. Are X, Y independent?

$\underline{X} = (X, Y)$



$f_X(x) = \int f_{X,Y}(x,y)\,dy > 0$ iff $x \in [0,3]$

$f_Y(y) > 0$ iff $y \in [0,4]$

No!

$$f_{(X,Y)}(x,y) = f_X(x) f_Y(y)$$

$$= \begin{cases} 1/6 & \text{on } \triangle \\ 0 & \text{off } \triangle \end{cases}$$

$\text{Support}(f_X) = \{x : f_X(x) \neq 0\} = [0,3]$

$\text{Supp}(f_Y) = \{y : f_Y(y) \neq 0\} = [0,4]$

$\therefore \text{supp } f_{X,Y} = [0,3] \times [0,4]$.

<u>Example</u>. Suppose $X \sim Exp(\lambda)$, $Y \sim Exp(\mu)$, and X, Y independent.

Find the distribution of $\min(X, Y)$.

$$F_{\min(X,Y)}(t) = \mathbb{P}(\min(X,Y) \leq t) = 1 - \mathbb{P}(\underbrace{\min(X,Y) > t}_{})$$

$$\{\min(X,Y) > t\} = \{X > t, Y > t\}$$

$$\therefore \mathbb{P}(X > t, Y > t) = \mathbb{P}(X > t)\,\mathbb{P}(Y > t) = e^{-\lambda t} e^{-\mu t} = e^{-(\lambda+\mu)t}$$

$$\therefore = 1 - e^{-(\lambda+\mu)t} = F_Z(t)$$

where $Z \sim Exp(\lambda+\mu)$

Theorem. Suppose $\underline{X} = (X_1, \dots, X_n)$ is a random vector, with $X_1, \dots, X_n$ independent.

For any functions $g_1, g_2, \dots, g_n : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}(g_1(X_1) g_2(X_2) \cdots g_n(X_n)) = \mathbb{E}(g_1(X_1)) \cdot \mathbb{E}(g_2(X_2)) \cdots \mathbb{E}(g_n(X_n))$$

Eg. X, Y indep.

$\mathbb{E}(X^2 Y)$

$= \mathbb{E}(X^2)\mathbb{E}(Y)$

Pf. ($n=2$, jointly continuous)

$$\mathbb{E}(g(X)h(Y)) = \iint_{\mathbb{R}^2} g(x) h(y) \underbrace{f_{(X,Y)}(x,y)}_{f_X(x) f_Y(y)} \, dx\, dy$$

$$= \int_{-\infty}^{\infty} g(x) f_X(x)\, dx \cdot \int_{-\infty}^{\infty} h(y) f_Y(y)\, dy = \mathbb{E}(g(X)) \cdot \mathbb{E}(h(Y)).$$

In particular, $\mathbb{E}(X_1 X_2 \cdots X_n) = \mathbb{E}(X_1)\mathbb{E}(X_2) \cdots \mathbb{E}(X_n)$.

Actually, the last theorem is iff.

↳ Suppose we know

$$\mathbb{E}(g_1(X_1)g_2(X_2)\cdots g_n(X_n)) = \mathbb{E}(g_1(X_1))\mathbb{E}(g_2(X_2))\cdots\mathbb{E}(g_n(X_n))$$

for all functions $g_1, g_2, \ldots, g_n$. Fix $A_1, \ldots, A_n \subseteq \mathbb{R}$.

Just take $g_1 = \mathbb{1}_{A_1} \cdots g_n = \mathbb{1}_{A_n}$

$$\mathbb{E}(g_1(X_1)) = \mathbb{E}(\mathbb{1}_{A_1}(X_1)) = \mathbb{P}(X_1 \in A_1).$$

$\leftarrow$ Ber(p) where $p = \mathbb{P}(X_1 \in A_1)$

$$g_1(X_1)g_2(X_2) = \mathbb{1}_{A_1}(X_1)\,\mathbb{1}_{A_2}(X_2)$$

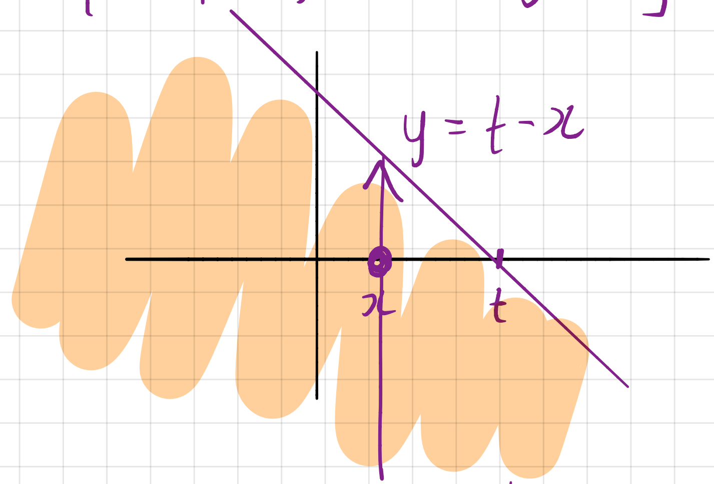
$$= \begin{cases} 0 & \text{if } X_1 \notin A_1 \\ 1 & \text{if } X_1 \in A_1 \end{cases} \cdot \begin{cases} 0 & \text{if } X_2 \notin A_2 \\ 1 & \text{if } X_2 \in A_2 \end{cases} = \begin{cases} 0 & \text{otherwise} \\ 1 & X_1 \in A_1 \ \& \ X_2 \in A_2 \end{cases}$$

$$\therefore \mathbb{E}(\text{" "}) = \mathbb{P}(X_1 \in A_1, X_2 \in A_2)$$

Convolution

Let X, Y be independent random variables. What can we say about the distribution of $X+Y$?

$T_t = \{(x,y) : x+y \le t\}$



$$F_{X+Y}(t) = \mathbb{P}(X+Y \le t)$$

$$= \mathbb{P}((X,Y) \in T_t)$$

$$= \iint_{T_t} f_{(X,Y)}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{t-x} f_{(X,Y)}(x,y)\,dy\,dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{t-x} f_X(x) f_Y(y)\,dy\,dx$$

$$= \int_{-\infty}^{\infty} f_X(x) \int_{-\infty}^{t-x} f_Y(y)\,dy$$

$$\therefore\ f_{X+Y}(t) = \frac{d}{dt}\left(\text{''} \quad \text{''}\right) = \int_{-\infty}^{\infty} f_X(x) f_Y(t-x)\,dx.$$

The convolution of two prob. densities is $(f*g)(t) = \int_{-\infty}^{\infty} f_X(x) f_Y(t-x)\,dx$

Eg. $X, Y \sim \mathrm{Exp}(\lambda)$, independent. Find f_{X+Y}.

$$f_{X+Y}(t) = f_X * f_Y(t) = \int_{-\infty}^{\infty} f_X(x) f_Y(t-x)\,dt$$

$$\lambda e^{-\lambda x} \mathbb{1}_{x \geq 0} \qquad \lambda e^{-\lambda(t-x)} \mathbb{1}_{t-x \geq 0}$$

$$0 \leq x \leq t$$

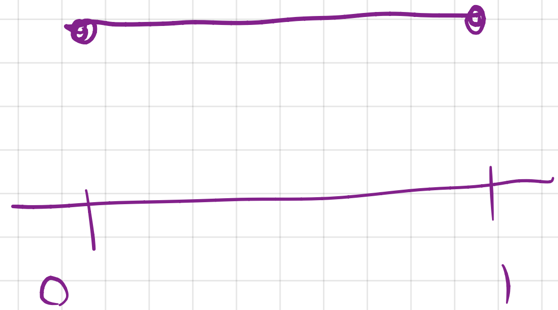
$X+Y \sim \Gamma(2, \lambda)$.

$$= \int_0^t \lambda \cancel{e^{-\lambda x}} \cdot \lambda \underbrace{e^{-\lambda(t-x)}}_{e^{-\lambda t} \cdot \cancel{e^{\lambda x}}} dx$$

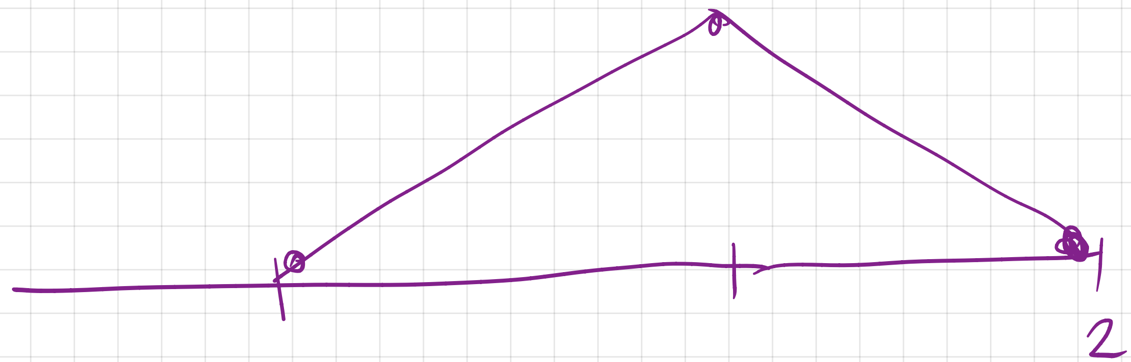
$$= \int_0^t \lambda^2 e^{-\lambda t}\,dx = \lambda^2 t e^{-\lambda t} \mathbb{1}_{t \geq 0}$$

E.g. $X, Y \sim \mathrm{Unif}([0,1])$, independent.

f_X, f_Y



f_{X+Y}



$$\mathbb{1}_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

E.g. $X \sim \mathrm{Unif}[0,1]$, $f_X(x) = \mathbb{1}_{[0,1]}(x)$