MATH 180A: Introduction to Probability

Lecture B00 (Nemish)

www.math.ucsd.edu/~ynemish/teaching/180a

Lecture C00 (Au)

www.math.ucsd.edu/~bau/f20.180a

Today: Covariance and correlation coefficient. MGF of sums of independent random variables

Video: Prof. Todd Kemp, Fall 2019

Next: ASV 9.3

Week 10:

- Homework 8 (due Friday, December 11, 11:59 PM)
- Quiz 5 on Wednesday, December 9 (lectures 17-20)
- CAPE at cape.ucsd.edu

Flashback to Math 20C

Given two vectors $\underline{v}, \underline{w}$ in $\mathbb{R}^n$, their dot product is

$$\underline{v}\cdot\underline{w} = \sum_{j=1}^{n} v_j w_j \qquad \underline{v}\cdot\underline{v} = \sum_{j=1}^{n} v_j^2$$

It is a positive bilinear form:

(1) $(a\underline{u}+b\underline{v})\cdot\underline{w} = a\,\underline{u}\cdot\underline{w} + b\,\underline{v}\cdot\underline{w}$ $\qquad (\underline{v}\cdot\underline{w} = \underline{w}\cdot\underline{v})$

(2) $\underline{v}\cdot(a\underline{u}+b\underline{w}) = a\,\underline{v}\cdot\underline{u} + b\,\underline{v}\cdot\underline{w}$

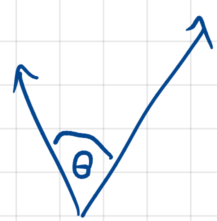
(3) $\underline{v}\cdot\underline{v} > 0$ unless $\underline{v}=\underline{0}$, in which case $\underline{v}\cdot\underline{v}=0$.

The length^2 of a vector is $\|\underline{v}\|^2 = \underline{v}\cdot\underline{v}$, $\quad$ length $= \|\underline{v}\| = \sqrt{\underline{v}\cdot\underline{v}}$

Two vectors are orthogonal if $\underline{v}\cdot\underline{w} = 0$

Cauchy-Schwarz Inequality: $|\underline{v}\cdot\underline{w}| \leq \|\underline{v}\|\,\|\underline{w}\|$

$$\frac{\underline{v}\cdot\underline{w}}{\|\underline{v}\|\,\|\underline{w}\|} = \cos\theta$$



$$-1 \leq \frac{\underline{v}\cdot\underline{w}}{\|\underline{v}\|\,\|\underline{w}\|} \leq 1$$

$-1 =$ iff $\underline{v} = a\underline{w}$ for $a<0$

$=1$ iff $\underline{v} = a\underline{w}$ for some $a>0$

The Geometry of Random Variables

$$\underline{\mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)}$$
$$\parallel$$

Dot product $\rightsquigarrow$ Covariance $\quad \mathrm{Cov}(X,Y) = \underline{\mathbb{E}\big((X-\mathbb{E}(X))(Y-\mathbb{E}(Y))\big)}.$

$\Downarrow$

(Almost) positive bilinear form

(1) $\mathrm{Cov}(aX_1+bX_2, Y) = a\,\mathrm{Cov}(X_1,Y) + b\,\mathrm{Cov}(X_2,Y)$

(2) $\mathrm{Cov}(X, aY_1+bY_2) = a\,\mathrm{Cov}(X,Y_1) + b\,\mathrm{Cov}(X,Y_2)$

(3) $\mathrm{Cov}(X,X) = \mathrm{Var}(X) > 0$ unless $X \equiv \text{const.}$ $(\mathrm{Var}(X) = 0)$

Orthogonal $\rightsquigarrow$ $\mathrm{Cov}(X,Y) = 0$ uncorrelated ($\Leftarrow$ independent)

Length^2 $\rightsquigarrow$ $\mathrm{Var}(X)$ $\quad$ Length $\rightsquigarrow \sqrt{\mathrm{Var}(X)} = \mathrm{S.D}(X)$

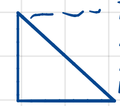
"Angles" $\rightsquigarrow$ $\dfrac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}} =: \mathrm{Corr}(X,Y)$

Theorem: $-1 \leq \mathrm{Corr}(X,Y) \leq 1$

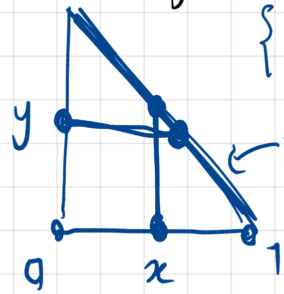
$\mathrm{Corr}(X,Y) = 0$ iff (X,Y) are uncorrelated $\not\Rightarrow$ independence

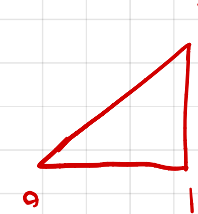
$\mathrm{Corr}(X,Y) = 1$ iff $Y = aX + b$ for some $a > 0$, $b \in \mathbb{R}$

$\mathrm{Corr}(X,Y) = -1$ iff $Y = aX + b$ for some $a < 0$, $b \in \mathbb{R}$.

Eg. (X,Y) uniformly distributed on the triangle  Area $= \frac{1}{2}$

$\{(x,y) : x,y \geq 0,\ x+y \leq 1\}$





$$f_{(X,Y)}(x,y) = 2\,\mathbb{1}_{\Delta}(x,y)$$

$$f_X(x) = \int_0^{1-x} f_{(X,Y)}(x,y)\,dy$$

$$= \int_0^{1-x} 2\,dy = 2(1-x)\,\mathbb{1}_{[0,1]}(x)$$

$$f_Y(x) = \int_0^{1-y} 2\,dx = 2(1-y)\,\mathbb{1}_{[0,1]}(y)$$

$X \sim Y$

$$\mathbb{E}(X) = \mathbb{E}(Y) = \int x f_X(x)\,dx = \int_0^1 x \cdot 2(1-x)\,dx = \frac{1}{3}$$

$$\mathbb{E}(X^2) = \mathbb{E}(Y^2) = \int x^2 f_X(x)\,dx = \int_0^1 x^2\, 2(1-x)\,dx = \frac{1}{6}.$$

$$\mathrm{Var}(X) = \mathrm{Var}(Y) = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{18}.$$

$$\mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = \iint_{\Delta} xy\, f_{(X,Y)}(x,y)\,dx\,dy - \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)$$

$$= \int_0^1 dx \int_0^{1-x} dy\ xy - \frac{1}{9} = \frac{1}{12} - \frac{1}{9} = \frac{-1}{36}.$$

$$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}X}\sqrt{\mathrm{Var}Y}} = \frac{-1/36}{\sqrt{\frac{1}{18}}\sqrt{\frac{1}{18}}} = \frac{-\frac{1}{36}}{\frac{1}{18}} = -\frac{1}{2}$$

Moment Generating Function Revisited

8.3

Suppose X, Y are independent, and $\boxed{M_X, M_Y < \infty}$ on an interval containing 0. Then

$$M_{X+Y}(t) = \mathbb{E}\left(e^{t(X+Y)}\right) = \mathbb{E}\left(\underset{\substack{\uparrow \\ \text{fn of } X}}{e^{tX}}\,\underset{\substack{\uparrow \\ \text{fn of } Y}}{e^{tY}}\right) = \mathbb{E}(e^{tX})\,\mathbb{E}(e^{tY}) = M_X(t) \cdot M_Y(t).$$

$\star$ If $M_Z(t) < \infty$ for all $t \in (-\delta, \delta)$, then M_Z determines the dist. of Z.

$\Rightarrow$ M_{X+Y} determines the dist. of $X+Y$.

E.g. $X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poisson}(\mu)$ independent

$$M_X(t) = e^{\lambda(e^t - 1)} \qquad M_Y(t) = e^{\mu(e^t-1)}$$

$$M_{X+Y}(t) = e^{\lambda(e^t-1)}\,e^{\mu(e^t-1)} = e^{\lambda(e^t-1)+\mu(e^t-1)} = e^{(\mu+\lambda)(e^t-1)}$$

$\uparrow$ the MGF of Poisson$(\lambda+\mu)$

$\Rightarrow X + Y \sim \text{Poisson}(\lambda+\mu)$

Eg. $X \sim N(\mu_1, \sigma_1^2) \qquad Y \sim N(\mu_2, \sigma_2^2)$ independent

$$Z \sim N(0,1), \quad M_Z(t) = e^{t^2/2}$$

$$X \sim \sigma_1 Z + \mu_1 \quad \therefore M_X(t) = M_{\sigma_1 Z + \mu_1}(t) = E(e^{t(\sigma_1 Z + \mu_1)})$$

$$= E(e^{(\sigma_1 t)Z} e^{\mu_1 t})$$

$$= e^{\mu_1 t} \underbrace{E(e^{(\sigma_1 t)Z})}_{M_Z(\sigma_1 t)} = e^{\mu_1 t} e^{(\sigma_1 t)^2/2}$$

$$= e^{\frac{\sigma_1^2}{2}t^2 + \mu_1 t}$$

$$\therefore M_{X+Y}(t) = e^{\frac{\sigma_1^2}{2}t^2 + \mu_1 t} \cdot e^{\frac{\sigma_2^2}{2}t^2 + \mu_2 t}$$

$$= e^{\frac{1}{2}(\sigma_1^2 + \sigma_2^2)t^2 + (\mu_1 + \mu_2)t} \Longleftarrow \text{Normal}$$

$$\simeq \text{MGF of } N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$