Math 180A: Introduction to Probability

Lecture B00 (Nemish)

math.ucsd.edu/~ynemish/teaching/180a

Lecture C00 (Au)

math.ucsd.edu/~bau/f20.180a

Today: ASV 9.2, 9.3

Video: Prof. Todd Kemp, Fall 2019

Next: Office hours on Friday

Week 10: Quiz 5 (Wednesday, Dec 9 on Lectures 17-20)

Homework 8 (due Friday, Dec 11)

Reminder: Chebyshev's Inequality

9.1

For any random variable X with finite

$$\mathbb{E}(X) = \mu \qquad Var(X) = \sigma^2$$

$$\boxed{\mathbb{P}(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}} \Rightarrow \mathbb{P}(|X-1000| \geq 3\cdot 15) \leq \frac{1}{3^2} = \frac{1}{9}$$

We proved this using the fact that $\mathbb{E}$ is monotone:

$$X \leq Y \Rightarrow \mathbb{E}(X) \leq \mathbb{E}(Y)$$

$= \sum_{k \geq 0}$ or $\int_0^\infty$

Pf. $\mathbb{E}(Y) - \mathbb{E}(X) = \mathbb{E}(Y) + \mathbb{E}(-X)$

$= \mathbb{E}(Y-X)$

$Y - X \geq 0$

$\therefore \mathbb{E}(Y-X) \geq 0$

Eg. Ramen Menya Ultra has, on average, 1000 customers/day, with a standard deviation of 15. Estimate the probability that today they will have between 956 and 1044 customers. #Customers $= X$

$$\mathbb{P}(956 \leq X \leq 1044) = \mathbb{P}(955 < X < 1045) = \mathbb{P}(|X-1000| < 3\cdot 15)$$

$$= \mathbb{P}(-45 < X-1000 < 45)$$

$$= 1 - \mathbb{P}(|X-1000| \geq 3\cdot 15) \geq 1 - \frac{1}{9} = \frac{8}{9}$$

$\approx 88.9\%$

(Weak) Law of Large Numbers

9.2

Let $X_1, X_2, X_3, \ldots, X_n, \ldots$ be an infinite sequence of i.i.d. random variables, each with $E(X_j) = \mu < \infty$ and $Var(X_j) = \sigma^2$ finite. $< \infty$

Let $S_n = X_1 + X_2 + \cdots + X_n$. Then for any fixed $\varepsilon > 0$,

$$\mathbb{P}\left(\left| \frac{S_n}{n} - \mu \right| \geq \varepsilon \right) = O\left(\frac{1}{n}\right)$$

Pf. Let $\bar{X}_n = S_n/n$.

$$E(\bar{X}_n) = E\left(\frac{S_n}{n}\right) = \frac{1}{n} E(X_1 + \cdots + X_n)$$
$$= \frac{1}{n}\left(E(X_1) + E(X_2) + \cdots + E(X_n)\right)$$
$$= \frac{1}{n}(\mu + \mu + \cdots + \mu)$$
$$= \mu.$$

$$\mathbb{P}(|\bar{X}_n - \mu| \geq \varepsilon)$$

want $\varepsilon = k\sqrt{Var(\bar{X}_n)} = k\sqrt{\frac{\sigma^2}{n}}$,

$$\| $$

$$\mathbb{P}\left(|\bar{X}_n - \mu| \geq k \cdot \sqrt{\frac{\sigma^2}{n}}\right) \leq \frac{1}{k^2}$$

$\therefore k = \varepsilon\sqrt{\frac{n}{\sigma^2}}$

$$\| $$

$$\frac{1}{\left(\varepsilon\sqrt{\frac{n}{\sigma^2}}\right)^2} = \frac{\sigma^2}{\varepsilon^2 n}$$

$\therefore S.D.(\bar{X}_n) = \sigma/\sqrt{n}$

$$Var(\bar{X}_n) = Var\left(\frac{1}{n}S_n\right) = \frac{1}{n^2} Var(S_n)$$
$$= \frac{1}{n^2} Var(X_1 + X_2 + \cdots + X_n)$$
$$= \frac{1}{n^2}\left(Var(X_1) + Var(X_2) + \cdots + Var(X_n)\right)$$
$$= \frac{1}{n^2}(\sigma^2 + \sigma^2 + \cdots + \sigma^2)$$

$$\| $$

$$\frac{1}{n^2} \cdot \sigma^2 n$$
$$= \frac{\sigma^2}{n}.$$

E.g. The Large Hadron Collider was built to detect and measure the mass of the Higgs Boson. Call the mass M.

For theoretical reasons, it is known that $M \leq 1.78 \times 10^{-23}$ g.

How many trials do the LHC physicists need to do to estimate the correct mass (via sample mean) within 10^{-24} g $= \varepsilon$, with probability $\geq 95\%$?

Trial measurements $M_1, M_2, M_3, \ldots, M_n, \ldots$ $\quad \bar{M}_n = \frac{M_1 + \cdots + M_n}{n}$

want $\mathbb{P}(|\bar{M}_n - \mu| < \varepsilon) \geq 95\%$

WLLN: $\mathbb{P}(|\bar{M}_n - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2} \cdot \frac{1}{n}$.

$$\therefore \mathbb{P}(|\bar{M}_n - \mu| < \varepsilon) \geq 1 - \frac{\sigma^2}{\varepsilon^2} \cdot \frac{1}{n} \overset{\text{want}}{\geq} 95\%$$

$$\text{i.e. } \frac{\sigma^2}{\varepsilon^2} \cdot \frac{1}{n} \leq 5\% \qquad n \geq \frac{\sigma^2}{\varepsilon^2} \cdot 20.$$

$$\geq \frac{(1.78 \times 10^{-23})^2}{(10^{-24})^2} \cdot 20$$

$$= 6337$$

Fact: (HW)

If $|X| \leq C$

$\Rightarrow \sqrt{\mathrm{var}(X)} \leq C$

Strong Law of Large Numbers

Let $X_1, X_2, X_3, \ldots, X_n, \ldots$ be an infinite sequence of i.i.d. random variables each with $\mathbb{E}(X_j) = \mu$.

Let $\bar{X}_n = \dfrac{X_1 + X_2 + \cdots + X_n}{n}$. Then

$$\mathbb{P}\left(\lim_{n\to\infty} \bar{X}_n = \mu\right) = 1,$$

Beautiful Proof uses existence of $\mathbb{E}(X_n^4) < \infty$, 9.3

The Law of Large Numbers says that if $X_1, X_2, X_3, \ldots$ are iid. with mean μ, and $S_n = X_1 + \cdots + X_n$, then

$$\frac{S_n}{n} \to \mu \qquad \text{i.e.} \qquad \frac{S_n - n\mu}{n} \to 0$$

$$\text{i.e.} \qquad S_n - n\mu = o(n).$$

At what exact rate n^α does $S_n - n\mu = O(n^\alpha)$?

Central Limit Theorem

Let $X_1, X_2, \ldots, X_n, \ldots$ be iid random variables with $E(X_j) = \mu$, $Var(X_j) = \sigma^2$.

Then $S_n - n\mu = O(\sqrt{n})$, and for $-\infty < a \leq b < \infty$,

$$\lim_{n\to\infty} P\Big(\underbrace{\frac{S_n - n\mu}{\sigma\sqrt{n}}} \in [a, b] \Big) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = \Phi(b) - \Phi(a).$$

$$\sigma\sqrt{n} = \sqrt{\sigma^2 n} = \sqrt{Var(S_n)} = \sqrt{Var(S_n - n\mu)}$$

Proof. Let $Y_n = (S_n - n\mu)/\sigma\sqrt{n}$. We will show that

$$M_{Y_n}(t) \longrightarrow M_{N(0,1)}(t) = e^{t^2/2} \quad \text{for all } t \in \mathbb{R}.$$

$$M_{Y_n}(t) = E(e^{tY_n}) = E\left(e^{t\cdot\frac{1}{\sigma\sqrt{n}}(S_n - n\mu)}\right)$$

$$= E\left(e^{\frac{t}{\sigma\sqrt{n}}(\mathring{X}_1 + \mathring{X}_2 + \dots + \mathring{X}_n)}\right)$$

$$= E\left(e^{\frac{t}{\sigma\sqrt{n}}\mathring{X}_1}\, e^{\frac{t}{\sigma\sqrt{n}}\mathring{X}_2} \cdots e^{\frac{t}{\sigma\sqrt{n}}\mathring{X}_n}\right)$$

independent.

$$= E\left(e^{\frac{t}{\sigma\sqrt{n}}\mathring{X}_1}\right) \cdots E\left(e^{\frac{t}{\sigma\sqrt{n}}\mathring{X}_n}\right)$$

$$= M_{\mathring{X}_1}\left(\frac{t}{\sigma\sqrt{n}}\right) \cdots M_{\mathring{X}_n}\left(\frac{t}{\sigma\sqrt{n}}\right)$$

$$= M_{\mathring{X}_1}(t/\sigma\sqrt{n})^n$$

$$S_n - n\mu = X_1 + X_2 + \dots + X_n - \mu - \mu - \dots - \mu = \mathring{X}_1 + \mathring{X}_2 + \dots + \mathring{X}_n$$

$$\mathring{X}_j = X_j - \mu.$$

i.d., $E(\mathring{X}_1) = 0$

$$E(\mathring{X}_1^2) = E((X_1-\mu)^2) = Var(X_1) = \sigma^2$$

$$M_{\mathring{X}_1}(s) = 1 + \cancel{\frac{E(\mathring{X}_1)}{1!}s} + \frac{E(\mathring{X}_1^2)}{2!}s^2 + \frac{E(\mathring{X}_1^3)}{3!}s^3 + \dots = 1 + \frac{\sigma^2}{2}s^2 + O(s^3)$$

$$M_{\mathring{X}_1}\left(\frac{t}{\sigma\sqrt{n}}\right) = 1 + \frac{\cancel{\sigma^2}}{2}\,\frac{t^2}{\cancel{\sigma^2} n} + O\left(\frac{1}{n^{3/2}}\right) \Rightarrow M_{\mathring{X}_1}\left(\frac{t}{\sigma\sqrt{n}}\right)^n = \left(1 + \frac{t^2}{2n} + O(n^{-3/2})\right)^n$$

$$= e^{t^2/2}$$