MATH 180A: Introduction to Probability

Lecture B00 (Nemish)

www.math.ucsd.edu/~ynemish/teaching/180a

Lecture C00 (Au)

www.math.ucsd.edu/~bau/f20.180a

Today: Independent trials

Next: ASV 3.3

Video: Prof. Todd Kemp, Fall 2019

Week 3:

- Homework 3 (due Friday October 23)
- Quiz 2 on Wednesday October 21
- Regrades for HW1: Mon, Oct 19 - Tue, Oct 20 (PST) on Gradescope

Independent Random Variables

2.3

A collection $X_1, X_2, X_3, \dots, X_n$ of random variables defined on the same sample space are independent if

for any $B_1, B_2, \dots, B_n \subseteq \mathbb{R}$, the events

$$\{X_1 \in B_1\}, \{X_2 \in B_2\}, \dots, \{X_n \in B_n\} \text{ are independent.}$$

E.g. ~~Ex~~. $\mathbb{P}(\{X_1 \in B_1\} \cap \{X_2 \in B_2\} \cap \cdots \cap \{X_n \in B_n\}) = \mathbb{P}(X_1 \in B_1)\mathbb{P}(X_2 \in B_2) \cdots \mathbb{P}(X_n \in B_n)$

Special Case: if the X_j are discrete random variables, it suffices to check the simpler condition

for any real numbers $t_1, t_2, \dots, t_n$

$$\Rightarrow \mathbb{P}(X_1 = t_1, X_2 = t_2, \dots, X_n = t_n) = \mathbb{P}(X_1 = t_1)\mathbb{P}(X_2 = t_2) \cdots \mathbb{P}(X_n = t_n)$$

E.g. Let $X_1, X_2, \dots, X_n$ be fair coin tosses. Denote $H \sim 1$, $T \sim 0$.

$$\mathbb{P}(X_1 = t_1, X_2 = t_2, \dots, X_n = t_n) = \frac{1}{2^n} = \underbrace{\frac{1}{2} \cdot \frac{1}{2} \cdots \frac{1}{2}}_{n} = \mathbb{P}(X_1 = t_1)\mathbb{P}(X_2 = t_2) \cdots \mathbb{P}(X_n = t_n)$$

Independent Trials

2.4

Experiments can have numerical observables, but sometimes you only observe whether there is success or failure

We model this with a random variable X taking value 1 with some probability p, and value 0 with probability $1-p$.

$$X \sim \mathrm{Ber}(p) \quad (\text{Bernoulli})$$

In practice, we usually repeat the experiment many times, making sure to use the same setup each trial. The previous trials do not influence the future ones.

$X_1, X_2, X_3, \ldots, X_n$ independent $\mathrm{Ber}(p)$ rv's. $(1-p)^4 p^2$.

$$\mathbb{P}(X_1 = 0, X_2 = 1, X_3 = 0, X_4 = 0, X_5 = 1, X_6 = 0) \quad (1-p)\,p\,(1-p)(1-p)\,p\,(1-p)$$

$$= \mathbb{P}(X_1 = 0)\,\mathbb{P}(X_2 = 1)\,\mathbb{P}(X_3 = 0)\,\mathbb{P}(X_4 = 0)\,\mathbb{P}(X_5 = 1)\,\mathbb{P}(X_6 = 0)$$

How many successful trials?

Run n independent trials, each with success probability p.

$$X_1, X_2, \ldots, X_n \sim \mathrm{Ber}(p).$$

Let $S_n = $ # successful trials

$$= X_1 + X_2 + X_3 + \cdots + X_n.$$

What is the distribution of S_n? $\quad S_n \in \{0, 1, 2, \ldots, n\}$

$$\mathbb{P}(S_n = k) = \mathbb{P}(\{\text{exactly } k \text{ of the } n \text{ trials are successful}\})$$

$0 \le k \le n$

$$= \binom{n}{k} p^k (1-p)^{n-k}$$

$p = \frac{1}{2}, \quad 1 - p = \frac{1}{2}$

$p^k (1-p)^{n-k} = \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} = \frac{1}{2^n}$

Binomial Distribution $\quad \mathrm{Bin}(n, p)$

$\mathrm{Ber}(p) = \mathrm{Bin}(1, p)$ | # heads in n fair coin tosses $\sim \mathrm{Bin}(n, \frac{1}{2})$

Eg. Roll a fair die 10 times. What is the probability that
success → 6 comes up at least 3 times?

$X_1, X_2, \dots, X_{10}$ indep. $\mathrm{Ber}(\frac{1}{6})$

$\therefore S_{10} = \mathrm{Bin}(10, \frac{1}{6})$

$$\mathbb{P}(S_{10} \geq 3) = \sum_{k=3}^{10} \mathbb{P}(S_{10}=k)$$

$$\frac{566299}{2519424} \doteq 22.5\%$$

$$= 1 - \mathbb{P}(S_{10} < 3) = 1 - \left(\mathbb{P}(S_{10}=0) + \mathbb{P}(S_{10}=1) + \mathbb{P}(S_{10}=2)\right)$$

$$= 1 - \left(\binom{10}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^{10} + \binom{10}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^9 + \binom{10}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^8 \right)$$

Eg. What is the probability that no 6 is rolled in the 10 rolls?

$$\mathbb{P}(S_{10}=0) = \binom{10}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^{10} = \left(\frac{5}{6}\right)^{10} \doteq 16\%$$

Now, keep rolling. Let N denote the first roll where a 6 appears. N is a random variable. What is its distribution?

First Success Time

N = first success in repeated independent trials (success rate p).

Model trials with (unlimited number) of independent $\mathrm{Ber}(p)$'s:

$X_1, X_2, X_3, X_4, \ldots$ $\qquad N \in \{1, 2, 3, \ldots\}$

$$\{N=k\} = \{X_1=0, X_2=0, X_3=0, \ldots, X_{k-1}=0, X_k=1\}$$

$$\mathbb{P}(N=k) = \mathbb{P}(\text{ " }) = \mathbb{P}(X_1=0)\,\mathbb{P}(X_2=0)\cdots\mathbb{P}(X_{k-1}=0)\,\mathbb{P}(X_k=1)$$

$$= (1-p)\cdot(1-p)\cdots(1-p)\cdot p = (1-p)^{k-1}p$$

Geometric Distribution $\mathrm{Geom}(p)$ on $\{0,1,2,3,\ldots\} = \mathbb{N}$.

$$\sum_{k=1}^{\infty}\mathbb{P}(N=k) = \sum_{k=1}^{\infty}(1-p)^{k-1}p = p\sum_{k=1}^{\infty}(1-p)^{k-1} = p\sum_{\ell=0}^{\infty}(1-p)^{\ell} = p\,\frac{1}{1-(1-p)} = \frac{p}{p} = 1.$$

$k-1=\ell$

geometric series

Rare Events

If $S_n = S_{n,p} \sim Bin(n,p)$, S_n is the number of successes in n independent trials each with success probability p.

What if p is very small, but n is very large?

One way to handle this mathematically is a **scaling limit**

↳ For each n, take $p \propto \frac{1}{n}$. $p = \frac{\lambda}{n}$ for some $\lambda > 0$.

$$\mathbb{P}(S_{n,p} = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$\lim_{n \to \infty} \left(\text{"} \qquad \text{"} \right) = ?$$

What happens as $n \to \infty$?

$$\mathbb{P}(S_{n,p}=k) = \binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}\frac{\lambda^k}{n^k}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-k}$$

$n \to \infty$

k is fixed

$$= \underset{=\,1}{\frac{n}{n}}\cdot\underset{\downarrow\,1}{\frac{n-1}{n}}\;\underset{\downarrow\,1}{\frac{n-2}{n}}\cdots\underset{\downarrow\,1}{\frac{n-k+1}{n}}\cdot\underset{=\,\frac{\lambda^k}{k!}}{\frac{\lambda^k}{k!}}\;\underbrace{\left(1-\frac{\lambda}{n}\right)^n}_{\downarrow\,e^{-\lambda}}\;\underbrace{\left(1-\underset{\downarrow\,0}{\frac{\lambda}{n}}\right)^{-k}}_{\downarrow\,1}$$

$$\left(1+\frac{x}{n}\right)^n \to e^x$$

$$\therefore \lim_{n\to\infty}\mathbb{P}(S_{n,\frac{\lambda}{n}}=k) = e^{-\lambda}\frac{\lambda^k}{k!}.$$

Poisson Distribution

A random variable X has the Poisson(λ) distribution if

$$\mathbb{P}(X=k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k=0,1,2,\dots$$

Check: $\displaystyle\sum_{k=0}^{\infty} \mathbb{P}(X=k) = 1.$

Summary

Sampling independent trials, the most important (discrete) probability distributions are:

- $\mathrm{Ber}(p)$: $\mathbb{P}(X=1)=p$, $\mathbb{P}(X=0)=1-p$ $\quad 0 \leq p \leq 1$

  (single trial with success probability p)

- $\mathrm{Bin}(n,p)$: $\mathbb{P}(S_n=k)=\binom{n}{k}p^k(1-p)^{n-k}$ $\quad 0 \leq k \leq n$

  (number of successes in n independent trials with rate p)

- $\mathrm{Geom}(p)$ $\quad \mathbb{P}(N=k)=(1-p)^{k-1}p$ $\quad k=0,1,2,\ldots$

  (first successful trial in repeated independent trials with rate p)

- $\mathrm{Poisson}(\lambda)$ $\quad \mathbb{P}(X=k)=e^{-\lambda}\frac{\lambda^k}{k!}$ $\quad k=0,1,2,\ldots$ $\quad \lambda>0$.

  (Approximates $\mathrm{Bin}(n,\lambda/n)$; number of rare events in many trials)