MATH 180A: Introduction to Probability

Lecture B00 (Nemish)

www.math.ucsd.edu/~ynemish/teaching/180a

Lecture C00 (Au)

www.math.ucsd.edu/~bau/f20.180a

Today: Expectation

Next: ASV 3.4

Video: Prof. Todd Kemp, Fall 2019

Week 3:

- Homework 3 (due Friday October 23)
- Midterm 1 next Wednesday, October 28, lectures 1 - 8

Poisson Distribution

A random variable X has the Poisson(λ) distribution if

$$\lim_{n\to\infty} \mathbb{P}(S_{n,\frac{\lambda}{n}}=k) = \mathbb{P}(X=k) = \underbrace{e^{-\lambda}\frac{\lambda^k}{k!}}_{\geq 0}, \quad k=0,1,2,\dots$$

$$\sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\underbrace{\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}} = e^{-\lambda}e^{\lambda} = 1 \quad \checkmark$$

Eg. A 100 year storm is a storm magnitude expected to occur in any given year with probability 1/100.

Over the course of a century, how likely is it to see at least 4 100 year storms?

$$\mathbb{P}(S_{100,\frac{1}{100}}\geq 4) = \sum_{k=4}^{100}\mathbb{P}(S_{100,\frac{1}{100}}=k) \approx \sum_{k=4}^{100} e^{-1}\frac{(1)^k}{k!}$$

$$\sum_{k=4}^{100}\binom{100}{k}\left(\frac{1}{100}\right)^k\left(1-\frac{1}{100}\right)^{100-k} \doteq 1.8374\% = 0.018374$$

$$\approx \sum_{k=4}^{\infty} e^{-1}\frac{1}{k!} = 1-\sum_{k=0}^{3} e^{-1}\frac{1}{k!} = 1.8988\%$$

Summary

Sampling independent trials, the most important (discrete) probability distributions are:

- $\mathrm{Ber}(p)$: $\quad \mathbb{P}(X=1)=p, \; \mathbb{P}(X=0)=1-p \qquad 0 \leq p \leq 1$

  (single trial with success probability p)

- $\mathrm{Bin}(n,p)$: $\quad \mathbb{P}(S_n=k)= \binom{n}{k} p^k (1-p)^{n-k} \qquad 0 \leq k \leq n$

  (number of successes in n independent trials with rate p)

- $\mathrm{Geom}(p)$ $\quad \mathbb{P}(N=k)=(1-p)^{k-1}p \qquad k=0,1,2,\ldots$

  (first successful trial in repeated independent trials with rate p)

- $\mathrm{Poisson}(\lambda)$ $\quad \mathbb{P}(X=k)=e^{-\lambda}\frac{\lambda^k}{k!} \qquad k=0,1,2,\ldots \qquad \lambda>0.$

  (Approximates $\mathrm{Bin}(n,\lambda/n)$; number of rare events in many trials)

Expectation

3.3

Toss a fair coin 1000 times, and record the sequence of outcomes.

1101000110110010011 . . .

Average them. $\frac{1}{1000}(1+1+0+1+0+0+0+1+1+0+1+1 \cdots)$

What size do you expect this number to have?

[About $\cancel{\tfrac{1}{2}}$ p of the outcomes are 1, about $\cancel{\tfrac{1}{2}}$ $1-p$ are 0. $\therefore$ Average $\approx \cancel{\tfrac{1}{2}(1)+\tfrac{1}{2}(0)}$ $= p(1)+(1-p)\cdot 0$

What if the coin is biased $\mathbb{P}(X_j=1)=p$, $\mathbb{P}(X_j=0)=1-p$.

Definition: Let X be a discrete random variable with possible values $t_1, t_2, t_3, \ldots$ The expectation or expected value of X is

$$E(X) := \sum_j t_j \, \mathbb{P}(X=t_j)$$

weighted average

"balance point"

Question: Is the expectation $\mathbb{E}(X)$ the value X is equal to most often?

(a) Yes, always.

(b) No, not generally.

Eg. Let X be the number rolled on a fair die. $X \in \{1,2,3,4,5,6\}$

$$\mathbb{E}(X) = \sum_{k=1}^{6} k \cdot \frac{1}{6} = \frac{1}{6}(1+2+3+4+5+6) = \frac{21}{6} = \frac{7}{2}.$$

Eg. Let Y be $\mathrm{Ber}(p)$. $\mathbb{E}(Y) = p \cdot 1 + (1-p) \cdot 0 = p$

Eg. You toss a biased coin (Y) repeatedly until the first heads. How long do you expect it to take?

N = the time the 1st heads comes up. $N \sim \mathrm{Geom}(p)$

$$\mathbb{E}(N) = \sum_{k=1}^{\infty} k \cdot \underbrace{\mathbb{P}(N=k)}_{(1-p)^{k-1}p} = p \sum_{k=1}^{\infty} k(1-p)^{k-1}$$

$$= p \cdot \frac{1}{(1-(1-p))^2} = \boxed{\frac{1}{p}}$$

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

$$\frac{d}{dx} \sum_{k=1}^{\infty} k x^{k-1} = \frac{1}{(1-x)^2}$$

E.g. $S_n \sim Bin(n,p)$ ($S_n = X_1 + X_2 + \cdots + X_n$ for X_j independent $Ber(p)$)

$$\mathbb{E}(S_n) = \sum_{k=0}^{n} k \cdot \mathbb{P}(S_n = k) = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k} = np$$

$$S_n = X_1 + X_2 + \cdots + X_n$$

$$\mathbb{E}(X_1 + \cdots + X_n) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \cdots + \mathbb{E}(X_n)$$

$$\mathbb{E}(X_j) = p$$

Eg. $X \sim Poisson(\lambda)$

$$\mathbb{E}(X) = \sum_{k=1}^{\infty} k \cdot e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} = e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^{j+1}}{j!} = \lambda e^{-\lambda} \underbrace{\sum_{j=0}^{\infty} \frac{\lambda^j}{j!}}_{e^{\lambda}} = \boxed{\lambda}$$

$k=0$

$j = k-1$

Eg. A factory has, on average, 3 accidents per month. Estimate the probability that there will be exactly 2 accidents this month.

$X = $ #accidents/month

$X \sim Poisson(3)$

$$\mathbb{E}(X) = 3 = \lambda \quad \Big\} \quad \mathbb{P}(X=2) = e^{-3} \frac{3^2}{2!} = 22.4\%$$

E.g. Toss a fair coin until tails comes up. If this is on the first toss, you win \$2 and stop. If heads comes up, the pot doubles, and you continue. That is, if the first tails is on the k^{th} toss, you win 2^k dollars.

What is your expected winnings?

$$W = \{ 2^k \text{ if the first tails is on the } k^{th} \text{ toss} \}$$

$$\mathbb{E}(W) = \sum_{k=1}^{\infty} 2^k \cdot \underbrace{\mathbb{P}(W=2^k)}_{\left(\frac{1}{2}\right)^{k-1} \cdot \frac{1}{2} = \frac{1}{2^k}} = \sum_{k=1}^{\infty} 2^k \cdot \frac{1}{2^k} = \infty.$$