1.

(a) This is a Bayes' rule calculation.

Let A = event that we selected urn II

B = event that first red is on third draw

Then

$$P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(A^c)P(B|A^c)}$$

$$= \frac{\frac{1}{2}\frac{6}{9}\frac{5}{8}\frac{3}{7}}{\frac{1}{2}\frac{6}{9}\frac{5}{8}\frac{3}{7} + \frac{1}{2}\frac{2}{9}\frac{1}{8}\frac{7}{7}}$$

1.

(b) This is also a Bayes' rule calculation, but with an obvious answer since urn I only has 2 green balls.

Let A = event that we selected urn II

B = event that first red is on fourth draw

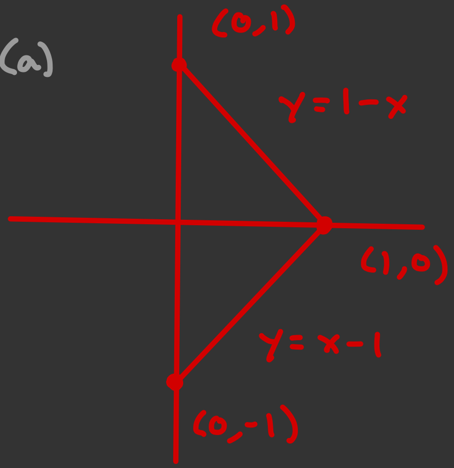
Then

$$P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(A^c)P(B|A^c)}$$

$$= \frac{\frac{1}{2}\frac{6}{9}\frac{5}{8}\frac{4}{7}\frac{3}{6}}{\frac{1}{2}\frac{6}{9}\frac{5}{8}\frac{4}{7}\frac{3}{6} + \frac{1}{2}0} = 1$$

2.

(a)



$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{\text{Area}(T)} = 1 & \text{if } (x,y) \in T \\ 0 & \text{if } (x,y) \notin T \end{cases}$$

$$\mathbb{E}[XY] = \int_0^1 \int_{x-1}^{1-x} xy \, dy\, dx = \int_0^1 x \frac{1}{2}\left(\overbrace{(1-x)^2 - (x-1)^2}^{\text{the same!}}\right) dx$$

$$= \int_0^1 0 \, dx = 0$$

again, the same

$$\mathbb{E}[Y] = \int_0^1 \int_{x-1}^{1-x} y \, dy\, dx = \int_0^1 \frac{1}{2}\left((1-x)^2 - (x-1)^2\right) dx = 0$$

for completeness, $\mathbb{E}[X] = \int_0^1 \int_{x-1}^{1-x} x \, dy \, dx$

$$= \int_0^1 (2-2x)x \, dx$$

$$= x^2 - \frac{2}{3}x^3 \Big|_0^1 = \frac{1}{3}$$

so, $\operatorname{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$

$$= 0 - \frac{1}{3} \cdot 0 = 0$$

2\.

(b) X and Y are not independent even though $Cov(X,Y) = 0$

To see this, we can calculate the marginal densities

$$f_X(x) = \int_{x-1}^{1-x} 1 \, dy = 2-2x \qquad \text{if } x \in [0,1]$$

$$f_Y(y) = \begin{cases} \int_0^{1-y} 1 \, dx = 1-y & \text{if } y \in [0,1] \\ \int_0^{y+1} 1 \, dx = y+1 & \text{if } y \in [-1,0) \end{cases}$$

Since $\mathcal{E}X \, \mathcal{E}Y \neq \mathcal{E}X,Y$, we conclude that

X and Y are not independent

3. $P(X>Y) = \int_{-1}^{1}\int_{y}^{\infty} f_{X,Y}(x,y)\, dx\, dy$

$$= \int_{-1}^{1}\int_{y}^{\infty} \frac{1}{2} e^{-x}\, \mathbb{1}\{x \geq 0\}\, dx\, dy$$

$$= \int_{-1}^{0}\int_{y}^{\infty} \frac{1}{2} e^{-x}\, \mathbb{1}\{x \geq 0\}\, dx\,dy + \int_{0}^{1}\int_{y}^{\infty} \frac{1}{2} e^{-x}\, \mathbb{1}\{x \geq 0\}\, dx\,dy$$

$$= \int_{-1}^{0}\int_{0}^{\infty} \frac{1}{2} e^{-x}\, dx\,dy + \int_{0}^{1}\int_{y}^{\infty} \frac{1}{2} e^{-x}\, dx\,dy$$

$$= \int_{-1}^{0} \frac{1}{2}\, dy + \int_{0}^{1} \frac{1}{2} e^{-y}\, dy = \frac{1}{2} + \left(-\frac{1}{2} e^{-y} \Big|_{0}^{1} \right) = 1 - \frac{1}{2} e^{-1}$$

4. Let E_i be the event that none of the rolls before the ith roll was a 6 and the ith roll and the i+1st roll match (and aren't equal to 6)

Then $X = \sum_{i=1}^{\infty} \mathbb{1}_{E_i}$ and

no 6 allowed

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} \mathbb{E}[\mathbb{1}_{E_i}] = \sum_{i=1}^{\infty} P(E_i) \qquad 5\tfrac{1}{6}\cdot\tfrac{1}{6}$$

$$= \sum_{i=1}^{\infty} \left(\frac{5}{6}\right)^{i-1} \left(\frac{5}{36}\right)$$

$$= \frac{5}{36} \sum_{i=0}^{\infty} \left(\frac{5}{6}\right)^{i} = \frac{5}{36} \frac{1}{1-\frac{5}{6}} = \frac{5}{6}$$

5.

(a) This is CLT after some rearranging.

Let $S_n = X_1 + \cdots + X_n$. Then we want

$$P(S_n \geq \sqrt{n} + n\mu) = P\left(\frac{S_n - n\mu}{\sqrt{4n}} \geq \frac{1}{\sqrt{4}}\right)$$

variance of X_i

By the CLT,

$$\lim_{n\to\infty} P\left(\frac{S_n - n\mu}{\sqrt{4n}} \geq \frac{1}{\sqrt{4}}\right) = 1 - \Phi\left(\frac{1}{\sqrt{4}}\right)$$

5.

(b) This is WLLN with some subtlety.

If $\mu > 0$, then $0 < \frac{\mu}{2} < \mu$. ①

If $\mu < 0$, then $0 > \frac{\mu}{2} > \mu$. ②

In ①, $P\left(S_n \geq \frac{n\mu}{2}\right) = P\left(\frac{S_n - n\mu}{n} \geq \frac{-\mu}{2}\right)$

$$\geq P\left(\left|\frac{S_n - n\mu}{n}\right| \leq \frac{\mu}{2}\right)$$

By the WLLN, $\lim_{n\to\infty} P\left(\left|\frac{S_n - n\mu}{n}\right| \leq \frac{\mu}{2}\right) = 1$

so $\lim_{n\to\infty} P\left(S_n \geq \frac{n\mu}{2}\right) = 1$

In 2 , $P\left(S_n \geq \frac{n\mu}{2}\right) = P\left(\frac{S_n - n\mu}{n} \geq \frac{-\mu}{2}\right)$

$$\leq P\left(\left|\frac{S_n - n\mu}{n}\right| \geq \frac{\mu}{2}\right)$$

By the WLLN, $\lim_{n\to\infty} P\left(\left|\frac{S_n - n\mu}{n}\right| \geq \frac{\mu}{2}\right) = 0$.

So, $\lim_{n\to\infty} P\left(S_n \geq \frac{n\mu}{2}\right) = 0$