MATH 180A - MIDTERM #1

FALL 2020

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT.

IF YOU DO NOT ASSIGN THE PAGES OF YOUR WORK TO THE QUESTIONS OF THE EXAM IN YOUR UPLOAD TO GRADESCOPE, YOU WILL LOSE POINTS (1 POINT FOR EVERY QUESTION THAT YOU FAIL TO ASSIGN THE PAGES TO).

WRITE YOUR NAME AND STUDENT ID ON THE FIRST PAGE OF YOUR UPLOAD

WRITE YOUR SOLUTIONS TO EACH PROBLEM ON SEPARATE PAGES. CLEARLY INDICATE AT THE TOP OF EACH PAGE THE NUMBER OF THE CORRESPONDING PROBLEM. DIFFERENT PARTS OF THE SAME PROBLEM CAN BE WRITTEN ON THE SAME PAGE (FOR EXAMPLE, PART (A) AND PART (B)).

FROM THE MOMENT YOU ACCESS THE MIDTERM ON GRADESCOPE, YOU WILL HAVE A TOTAL OF 65 MINUTES TO COMPLETE AND UPLOAD YOUR SOLUTIONS TO GRADESCOPE. THE EXAM IS WRITTEN TO TAKE 50 MINUTES. IT IS YOUR RESPONSIBILITY TO UPLOAD YOUR SOLUTIONS ON TIME.

THE MIDTERM IS OPEN TEXTBOOK, AND YOU CAN REWATCH LECTURES AND REVIEW THE LECTURE SLIDES. YOU ARE NOT ALLOWED TO USE CALCULATORS, COMPUTER ALGEBRA SYSTEMS, OR SIMILAR SOFTWARE. YOU ARE NOT ALLOWED TO CONSULT OTHER PEOPLE OR RESOURCES ON THE INTERNET. IN PARTICULAR, CONTRACT CHEATING AND OTHER INSTANCES OF ACADEMIC DISHONESTY CAN RESULT IN SUSPENSION OR DISMISSAL.

EXCEL WITH INTEGRITY PLEDGE

I pledge to be fair to my classmates and instructors by completing all of my academic work with integrity. This means that I will respect the standards set by the instructor and institution, be responsible for the consequences of my choices, honestly represent my knowledge and abilities, and be a community member that others can trust to do the right thing even when no one is watching. I will always put learning before grades, and integrity before performance. I pledge to excel with integrity.

In addition to the above, I pledge that I did not receive outside assistance with this exam. Outside assistance includes but is not limited to other people, the internet, and resources beyond the textbook, lecture notes/videos, and homework assignments.

To acknowledge that you agree to this pledge, you must copy the sentence:

> I choose to excel with integrity as a member of the University of California, San Diego.

Make sure to sign your name below it and date it as well. Exams without this pledge will *not* be graded.

1. (25 points) Suppose we have an urn with 10 red balls, 15 blue balls, and 20 green balls. Consider the following experiment: we draw a ball uniformly at random, record its color, and put it back into the urn. We perform the experiment 10 times.

(a) (10 points) What is the probability that we see a red ball exactly 3 times?

Solution. This is asking for $P(X = 3)$ where $X \sim \text{Bin}(10, \frac{10}{10+15+20})$. So,

$$P(X = 3) = \binom{10}{3} \left(\frac{10}{10 + 15 + 20}\right)^3 \left(1 - \frac{10}{10 + 15 + 20}\right)^7$$

□

(b) (15 points) What is the probability that the color of the ball in the first experiment is different from the color of the ball in the last experiment?

Solution. The easiest way to compute this is to take one minus the probability of the complement:

$$1 - \left(\frac{10}{10 + 15 + 20}\frac{10}{10 + 15 + 20} + \frac{15}{10 + 15 + 20}\frac{15}{10 + 15 + 20} + \frac{20}{10 + 15 + 20}\frac{20}{10 + 15 + 20}\right)$$

□

2. (25 points) Suppose we have an urn with 10 red balls, 15 blue balls, and 20 green balls. We draw three balls without replacement uniformly at random.

(a) (5 points) What is the probability that we have more red balls than non-red balls?

Solution. The two possibilities we need to account for are {2 red and 1 non-red} and {3 red and 0 non-red}. So, the probability is

$$\frac{\binom{10}{2}\binom{35}{1}}{\binom{45}{3}} + \frac{\binom{10}{3}\binom{35}{0}}{\binom{45}{3}}$$

□

(b) (10 points) What is the probability that we have more red balls than green balls?

Solution. The *three* possibilities we need to account for are {1 red, 2 blue}, {2 red, 1 non-red}, {3 red, 0 non-red}. So, the probability is

$$\frac{\binom{10}{1}\binom{15}{2}\binom{20}{0}}{\binom{45}{3}} + \frac{\binom{10}{2}\binom{35}{1}}{\binom{45}{3}} + \frac{\binom{10}{3}\binom{35}{0}}{\binom{45}{3}}$$

□

(c) (10 points) What is the probability that at least two of the three balls have the same color?

Solution. The easiest way to compute this is to take one minus the probability of the complement. The complement of at least two of the three balls having the same color is no two balls have the same color. Since we only have three colors and we are drawing three balls, this means that we must have 1 red, 1 blue, and 1 green. So, the probability is

$$1 - \frac{\binom{10}{1}\binom{15}{1}\binom{20}{1}}{\binom{45}{3}}$$

□

3. (25 points) Suppose that we have three fair and independent dice: the first die is 6-sided, the second die is k-sided, and the third die is $2k$-sided. Assume that k is a whole number with $k > 6$. You put the k-sided die and the $2k$-sided die in a bag. Your friend picks one of the dice out of this bag uniformly at random and rolls both it and the 6-sided die. Your friend then tells you that the numbers are the same. What is the probability that your friend drew the k-sided die from the bag? You should simplify your answer to this question to receive full credit.

Solution. Let X be the outcome from rolling the 6-sided die and Y the outcome from rolling the die drawn from the bag. Let

$$A = \{k\text{-sided die was drawn from the bag}\};$$
$$A^c = \{2k\text{-sided die was drawn from the bag}\}.$$

Then we want to compute

$$\begin{aligned} P(A|X = Y) &= \frac{P(A \cap \{X = Y\})}{P(X = Y)} \\ &= \frac{P(X = Y|A)P(A)}{P(X = Y|A)P(A) + P(X = Y|A^c)P(A^c)}. \end{aligned}$$

Note that $P(A) = P(A^c) = \frac{1}{2}$. Moreover, since $k > 6$,

$$P(X = Y|A) = \sum_{i=1}^{6} \frac{1}{6}\frac{1}{k} = \frac{1}{k};$$
$$P(X = Y|A^c) = \sum_{i=1}^{6} \frac{1}{6}\frac{1}{2k} = \frac{1}{2k}.$$

So, our answer simplifies to

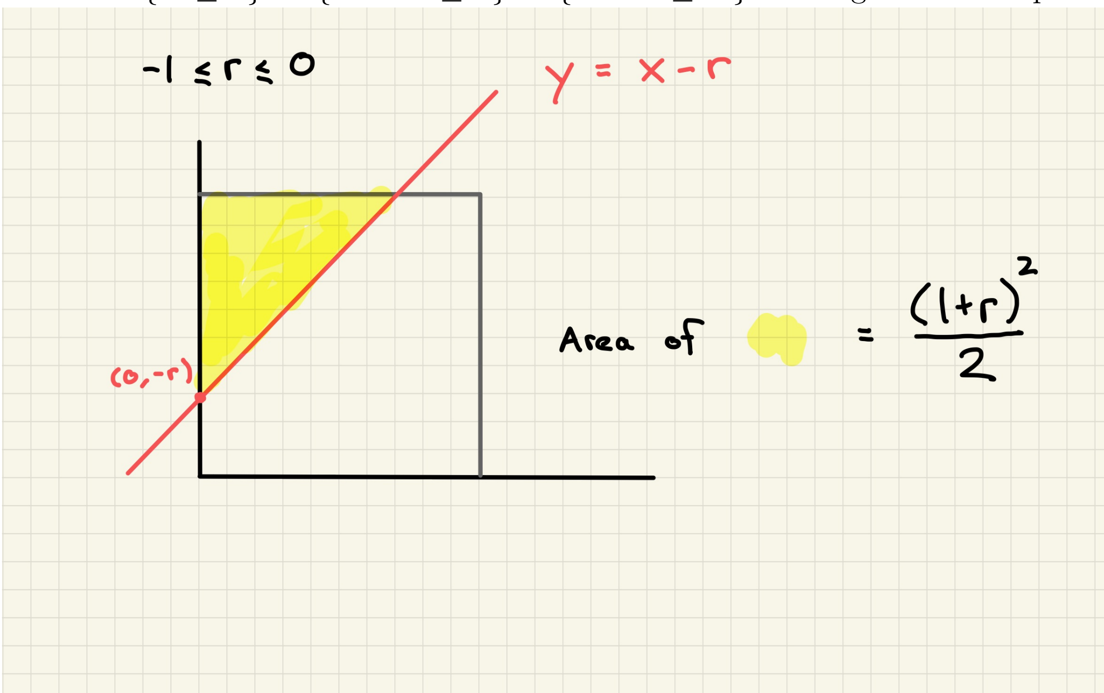
$$\begin{aligned} P(A|X = Y) &= \frac{P(X = Y|A)P(A)}{P(X = Y|A)P(A) + P(X = Y|A^c)P(A^c)} \\ &= \frac{\frac{1}{k}\frac{1}{2}}{\frac{1}{k}\frac{1}{2} + \frac{1}{2k}\frac{1}{2}} \\ &= \frac{2}{3}. \end{aligned}$$

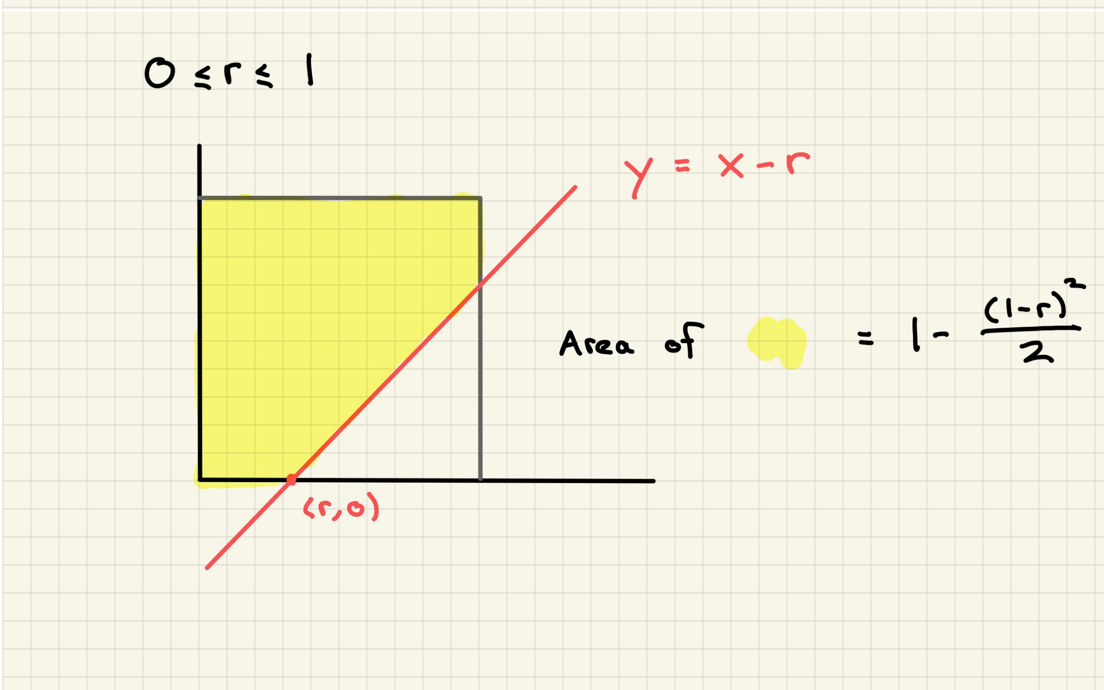
$\square$

4. (25 points) Suppose we have a square in $\mathbb{R}^2$ with vertices $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$. Let (X,Y) denote a uniformly chosen random point inside this square.

(a) (20 points) Let $Z = X - Y$. Find the CDF of Z. (Hint: draw a picture)

Solution. Even before drawing a picture, we can determine some of the values of the CDF of Z. Note that $X - Y \in [-1,1]$. So, $F_Z(r) = 0$ for $r < -1$ and $F_Z(r) = 1$ for $r > 1$. It remains to compute $F_Z(r)$ for $r \in [-1,1]$. We rewrite the event $\{Z \le r\}$ as $\{X - Y \le r\} = \{X - r \le Y\}$. This gives us two pictures:





So,

$$F_Z(r) = \begin{cases} 0 & \text{if } r < -1; \\ \frac{(1+r)^2}{2} & \text{if } -1 \leq r \leq 0; \\ 1 - \frac{(1-r)^2}{2} & \text{if } 0 \leq r \leq 1; \\ 1 & \text{if } r > 1. \end{cases}$$

□

(b) (5 points) Determine if Z is a continuous random variable, a discrete random variable, or neither. If continuous, determine the probability density function of Z. If discrete, determine the probability mass function of Z. If neither, explain why.

Solution. Our CDF is continuous everywhere and piecewise differentiable. So, the random variable is continuous. A probability density is given by

$$f_Z(r) = \begin{cases} 0 & \text{if } r < -1; \\ 1 + r & \text{if } -1 \leq r < 0; \\ 1 - r & \text{if } 0 \leq r \leq 1; \\ 0 & \text{if } r > 1. \end{cases}$$

□