MATH 180A - INTRODUCTION TO PROBABILITY
PRACTICE MIDTERM #1 ADDITIONAL PROBLEMS

FALL 2020

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REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT.

1. At a political meeting there are 7 progressives and 7 conservatives. We choose five people uniformly at random to form a committee (president, vice-president and 3 regular members).

(a) Let A be the event that we end up with more conservatives than progressives. What is the probability of A?

(b) Let B be the event that Ronald, representing conservatives, becomes the president, and Felix, representing liberals, becomes the vice-president. What is the probability of B?

Solution.

(a) For $0 \le i \le 5$ define the events $A_i = \{i \text{ conservatives on the committee}\}$. Then

$$P(A_i) = \frac{\binom{7}{i}\binom{7}{5-i}}{\binom{14}{5}}.$$

If we now define $B = \{\text{more coservetives than progressives}\}$, then $B = A_3 \cup A_4 \cup A_5$. We can now compute

$$P(B) = \frac{\binom{7}{4}\binom{7}{2} + \binom{7}{4}\binom{7}{1} + \binom{7}{5}\binom{7}{0}}{\binom{14}{5}}$$

(b) Define $C = \{\text{Ronald president, Felix vice-president}\}$. Then

$$P(C) = \frac{\binom{12}{3}}{\binom{14}{5} \cdot 5 \cdot 4} = \frac{1}{13 \cdot 14}.$$

2. Let A, B be events in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

(a) Suppose that A, B satisfy $A \cap B = \emptyset$. If A and B are independent, what can you say about $\mathbb{P}(A)$ and $\mathbb{P}(B)$?

(b) Suppose that $\mathbb{P}(A) = 0.5$ and $\mathbb{P}(B) = 0.8$. What possible range of values can $\mathbb{P}(A \cap B)$ have?

Solution.

(a) If A and B are independent, then

$$P(A \cap B) = P(A)P(B) = P(\emptyset) = 0,$$

which implies that either $P(A) = 0$ or $P(B) = 0$.

(b) By the definition of probability, we have $0 \le P(A \cup B) \le 1$. Applying the inclusion-exclusion formula gives

$$0 \le P(A \cup B) = P(A) + P(B) - P(A \cap B) \le 1,$$

which yields that $P(A \cap B) \ge P(A) + P(B) - 1 = 0.3$. On the other hand

$$P(A \cap B) \le P(A) = 0.5, \quad P(A \cap B) \le P(B) = 0.8,$$

so $0.3 \le P(A \cap B) \le 0.5$.

3. Suppose that we have two possibly unfair coins: the first coin takes heads with probability $p \in (0, 1)$ and tails with probability $1 - p$; the second coin takes heads with probability

$q \in (0,1)$ and tails with probability $1-q$. The coins are independent of each other and consecutive flips are also independent.

Consider the following game. Flip both coins simultaneously. If the coins land on the same side (for example, both land on heads), then you win. Otherwise, then you lose. Let X be the number of times you play the game until you win. For example, X is equal to 1 if you win on your first play of the game. Determine the probability mass function of X.

Solution. The probability of winning in one game is given by

$$P((H,H)) + P((T,T)) = pq + (1-p)(1-q) = 1 + 2pq - p - q.$$

Denote this number by $s := 1 + 2pq - p - q$. Then X has geometric distribution with parameter s, $X \sim \text{Geom}(s)$, and

$$P(X = k) = (1-s)^{k-1}s = (p + q - 2pq)^{k-1}(1 + 2pq - p - q).$$

4. Consider a point $P = (X, Y)$ chosen uniformly at random inside of the unit square $[0,1]^2 = [0,1] \times [0,1] = \{(x,y) : 0 \leq x, y \leq 1\}$. Let $Z = \min(X,Y)$ be the random variable defined as the minimum of the two coordinates of the point. For example, if $P = (\frac{1}{2}, \frac{1}{3})$, then $Z = \min(\frac{1}{2}, \frac{1}{3}) = \frac{1}{3}$. Determine the cumulative distribution function of Z. Determine if Z is a continuous random variable, a discrete random variable, or neither. If continuous, determine the probability density function of Z. If discrete, determine the probability mass function of Z. If neither, explain why.

(Hint: Draw a picture.)

Solution. First notice that

$$\{\min(X,Y) \leq s\} = \{X \leq s \text{ or } Y \leq s\} = \{X \leq s\} \cup \{Y \leq s\}.$$

For any $s \in (0,1)$ denote $A_s := \{(x,y) \in [0,1]^2 \mid x \leq s \text{ or } y \leq s\}$. Then

$$\text{Area}(A_s) = 1 - (1-s)^2 = 2s - s^2.$$

Since the point P is chosen uniformly at random, for $s \in (0,1)$

$$\begin{aligned} F_Z &= P(\{\min(X,Y) \leq s\}) \\ &= \text{Area}(A_s) \\ &= 1 - (1-s)^2 = 2s - s^2. \end{aligned}$$

Now the cumulative distribution function (CDF) is given by

$$F_Z(s) = \begin{cases} 0, & s < 0, \\ 2s - s^2, & 0 \leq s < 1, \\ 1, & s \geq 1. \end{cases}$$

The CDF is continuous, so we can compute the PDF

$$f_Z(s) = \begin{cases} 0, & s < 0, \\ 2 - 2s, & 0 \leq s < 1, \\ 0, & s \geq 1. \end{cases}$$

5. You shoot an arrow (uniformly at random) at a round target of radius 50 cm. If you hit a point at a distance ≤ 10 cm from the center of the target, you are awarded 10 points; if the point you hit is between 10 and 20 cm from the center, you get 5 point; if the point you hit is between 20 and 30 cm from the center, you get 3 points; if the point you hit is between 30 and 40 points from the center you get 1 point; if you hit a point which is 40 cm or more from the center you get 0 points. Let X be a random variable that gives the number of points you get after *one* shot.

(a) Is the random variable X continuous, discrete, neither or both?
(b) If X is continuous or discrete, compute the p.m.f./p.d.f. of X.
(c) Compute and plot the c.d.f. of the random variable X.

Solution.

(a) Random variable X takes values in a finite set $\{0, 1, 3, 5, 10\}$, therefore X is a discrete random variable.
(b) Since we shoot the arrow uniformly at random, the probability $P(X = i)$ for $i \in \{0, 1, 3, 5, 10\}$ is equal to ratio between the area of the annulus (disc) giving i points and the area of the whole target. For example,

$$P(X = 10) = \frac{\pi 10^2}{\pi 50^2} = \frac{1}{25},$$
$$P(X = 5) = \frac{\pi 20^2 - \pi 10^2}{\pi 50^2} = \frac{3}{25},$$

and so on. The PMF of the random variable X is then given by

i	0	1	3	5	10
$P(X = i)$	9/25	7/25	5/25	3/25	1/25

(c) From the above table we can compute the CDF of the random variable X

$$F_X(s) = \begin{cases} 0, & s < 0 \\ 9/25, & 0 \leq s < 1 \\ 16/25, & 1 \leq s < 3 \\ 21/25, & 3 \leq s < 5 \\ 24/25, & 5 \leq s < 10 \\ 1, & s \geq 10. \end{cases}$$