MATH 180A - INTRODUCTION TO PROBABILITY
PRACTICE MIDTERM #1

FALL 2020

Name (Last, First): ____________________

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REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT.

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

(a) Suppose that $A, B \in \mathcal{F}$ satisfy

$$\mathbb{P}(A) + \mathbb{P}(B) > 1.$$

Making no further assumptions on A and B, prove that $A \cap B \neq \emptyset$.

Solution. First method (proof by contradiction). Assume that $A \cap B = \emptyset$. Then $P(A \cup B) = P(A) + P(B) > 1$, contradiction. Therefore, $A \cap B \neq 0$.

Second method.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq 1,$$

therefore $P(A \cap B) \geq P(A) + P(B) - 1 > 0$, and we conclude that $A \cap B \neq \emptyset$.

(b) Prove that A is independent from itself if and only if $\mathbb{P}(A) \in \{0, 1\}$.

Solution. By definition, events A and B are independent if $P(A \cap B) = P(A)P(B)$. Take $A = B$, so that A being independent of A is equivalent to $P(A \cap A) = P(A)P(A)$. Since $A \cap A = A$, we have that $P(A)$ satisfies

$$(P(A))^2 = P(A).$$

The only two numbers that satisfy the above equation are numbers 0 and 1.

2. Roll two fair dice repeatedly. If the sum is ≥ 10, then you win.

(a) What is the probability that you start by winning 3 times in a row?

Solution. Define an event $A := \{\text{roll 2 dice and get the sum} \geq 10\}$ and denote $p = P(\text{Success}) = P(A)$, the probability of winning in one single game. Now, we compute p

$$p = \frac{1}{36} \#\{(4,6), (5,5), (6,4), (5,6), (6,5), (6,6)\} = \frac{1}{6}. \tag{1}$$

The probability that you observe the successful outcome (win) three times in a row is

$$p^3 = \frac{1}{6^3}.$$

You can also think about it in terms of a random variable X with binomial distribution with parameters 3 and $1/6$, $X \sim \text{Bin}(3, 1/6)$ that counts the number of wins in 3 games, and the probability that you win three times is

$$P(X = 3) = \binom{3}{3} p^3 (1-p)^0 = \frac{1}{6^3}.$$

(b) What is the probability that after rolling the pair of dice 5 times you win exactly 3 times?

Solution. Using the same notation as in question (a), let Y be a random variable having binomial distribution with parameters 5 and p representing the numbers of wins in 5 games. Then the probability that you win 3 out of 5 games is

$$\begin{aligned} P(Y=3) &= \binom{5}{3} p^3 (1-p)^2 \\ &= \binom{5}{3} \frac{1}{6^3} \left(\frac{5}{6}\right)^2 \\ &= \frac{250}{6^5}. \end{aligned}$$

(c) What is the probability that the first time you win is before the tenth roll (of the pair), but after the fifth?

Solution. This problem is about the first successful experiment in a sequence of independent trials with success rate $p = 1/6$. Define a random variable $Z \sim \text{Geom}(1/6)$, the random variable that represents the first winning game. Then the probability that the first win occurs before the tenth roll but after the fifth is

$$\begin{aligned} P(5 < Z < 10) &= P(Z=6) + P(Z=7) + P(Z=8) = P(Z=9) \\ &= p(1-p)^5 + p(1-p)^6 + p(1-p)^7 + p(1-p)^8 \\ &= p(1-p)^5(1 + (1-p) + (1-p)^2 + (1-p)^3) \\ &= p(1-p)^5 \frac{1-(1-p)^4}{1-(1-p)} = (1-p)^5 - (1-p)^9. \end{aligned}$$

Think about the meaning of the quantity $(1-p)^5 - (1-p)^9$.

3. A box contains 3 coins, two of which are fair and the third has probability 3/4 of coming up heads. A coin is chosen randomly from the box and tossed 3 times.

(a) What is the probability that all 3 tosses are heads?

Solution. Define the following events:

$$\begin{aligned} A &= \{\text{all 3 tosses are heads}\} \\ B_1 &= \{\text{chosen coin is fair}\} \\ B_2 &= \{\text{chosen coin is biased}\}. \end{aligned}$$

Then B_1 and B_2 form a partition of the sample space with

$$P(B_1) = \frac{2}{3}, \qquad P(B_2) = \frac{1}{3},$$

and depending on which coin was chosen from the box, we have the conditional probabilities of observing heads on all three tosses

$$P(A|B_1) = \left(\frac{1}{2}\right)^3, \qquad P(A|B_2) = \left(\frac{3}{4}\right)^3.$$

Then we can compute $P(A)$ using the law of total probability

$$\begin{aligned} P(A) &= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) \\ &= \frac{1}{8} \cdot \frac{2}{3} + \left(\frac{3}{4}\right)^3 \frac{1}{3} \\ &= \frac{43}{192}. \end{aligned}$$

(b) Given that all three tosses are heads, what is the probability that the biased coin was chosen?

Solution. Using the Bayes' rule, we have

$$P(B_2|A) = \frac{P(A|B_2)P(B_2)}{P(A)} = \frac{27}{43}.$$

4. Let X be a discrete random variable taking the values $\{1, 2, \ldots, n\}$ all with equal probability. Let Y be another discrete random variable taking values in $\{1, 2, \ldots, n\}$. Assume that X and Y are independent. Show that $\mathbb{P}(X = Y) = \frac{1}{n}$. (Hint: you do not need to know the distribution of Y to calculate this.)

Solution. Since both random variables X and Y take values in the set $\{1, 2, \ldots, n\}$, the event $\{X = Y\}$ can be written as a disjoint union

$$\{X = Y\} = \cup_{k=1}^{n} \{X = k, Y = k\},$$

therefore the probability of $\{X = Y\}$ is equal to

$$P(X = Y) = \sum_{k=1}^{n} P(X = k, Y = k).$$

From the independence of X and Y and the fact that $P(X = k) = 1/n$ for all $k \in \{1, \ldots, n\}$ we have that

$$\begin{aligned} \sum_{k=1}^{n} P(X = k, Y = k) &= \sum_{k=1}^{n} P(X = k)(Y = k). \\ &= \frac{1}{n} \sum_{k=1}^{n} P(Y = k). \end{aligned}$$

Since Y is distributed on $\{1, \ldots, n\}$, $\sum_{k=1}^{n} P(Y = k) = 1$ and we conclude that $P(X = Y) = 1/n$.

5. Consider a point $P = (X, Y)$ chosen uniformly at random inside of the triangle in $\mathbb{R}^2$ that has vertices $(1, 0)$, $(0, 1)$, and $(0, 0)$. Let $Z = \max(X, Y)$ be the random variable defined as the maximum of the two coordinates of the point. For example, if $P = (\frac{1}{2}, \frac{1}{3})$, then $Z = \max(X, Y) = \frac{1}{2}$. Determine the cumulative distribution function of Z. Determine if Z is a continuous random variable, a discrete random variable, or neither. If continuous, determine the probability density function of Z. If discrete, determine the probability mass function of Z. If neither, explain why.

(Hint: Draw a picture.)

Solution.

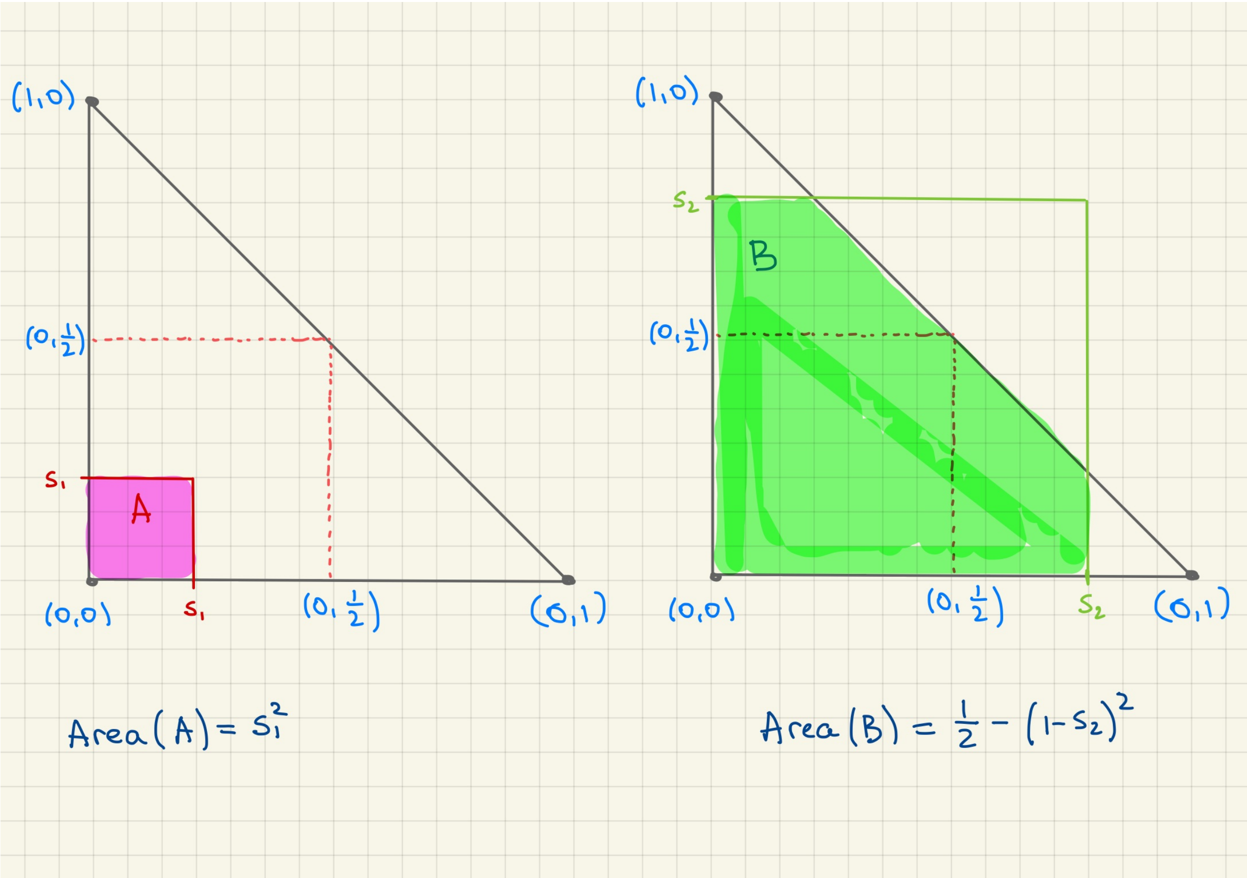
First note, that since $Z = \max\{X, Y\}$, the event $\{Z \le s\}$ can be rewritten as

$$\{\max\{X, Y\} \le s\} = \{X \le s, Y \le s\}.$$

So in order to compute the CDF of Z, we have to compute the probability that $X \le s, Y \le s$ for all $s \in \mathbb{R}$, where (X, Y) is uniformly chosen from the triangle with vertices $(0, 0)$, $(0, 1)$ and $(1, 0)$.

It is clear that $F_Z(s) = P(Z \le s) = 0$ if $s \le 0$ and $F_Z(s) = 1$ if $s \ge 1$.

For $s \in (0, 1)$ two situations are possible (see the picture below)



If $0 \le s \le 1/2$, then

$$P(X \le s, Y \le s) = \frac{\text{Area}(A)}{\text{Area of the triagle}} = \frac{s^2}{\frac{1}{2}} = 2s^2.$$

If $1/2 \le s \le 1$, then

$$P(X \le s, Y \le s) = \frac{\text{Area}(B)}{\text{Area of the triagle}} = \frac{\frac{1}{2} - (1-s)^2}{\frac{1}{2}} = 1 - 2(1-s)^2.$$

We finally get that

$$F_Z(s) = \begin{cases} 0, & s \leq 0, \\ 2s^2, & 0 \leq s \leq 1/2, \\ 1 - 2(1-s)^2, & 1/2 \leq s \leq 1, \\ 1, & s \geq 0. \end{cases}$$

It is clear that the function is continuous at points $s = 0$ and $s = 1$, and we can check that it is also continuous at $s = 1/2$

$$2\left(\frac{1}{2}\right)^2 = \frac{1}{2} = 1 - 2\left(1 - \frac{1}{2}\right)^2.$$

Since the CDF of Z is continuous, the random variable Z is a continuous random variable. In order to compute its probability density function, differentiate the CDF

$$f_Z(s) = \begin{cases} 0, & s \leq 0, \\ 4s, & 0 \leq s \leq 1/2, \\ 4 - 4s, & 1/2 \leq s \leq 1, \\ 0, & s \geq 0. \end{cases}$$